Value Shares of Technologically Complex Products

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[1]Competition Dynamics. Earlier drafts benefited from discussions with Peter Boberg, Peter Schwechheimer and Andrew Tepperman, but I alone am responsible for any errors. I received valuable research assistance from Matt Johnson.

[2]The ideas presented in this paper reflect some of the factors that may contribute to a complete patent damages analysis. Nothing in this paper should be construed as a complete damages analysis, which in general depends on additional facts that are specific to any given case.

Abstract

Although patent law requires the apportionment of profit between an asserted patent and other inputs, the typical analysis need not (and often demonstrably does not) satisfy a simple adding-up constraint on the value of the apportioned inputs. This problem is especially acute for "complex" products that may embody hundreds of patents. Based on a wide range of studies of the patent value distribution, I present simple but robust guidelines for determining the share of value contributed by any given patent, and apply them in several familiar contexts. I find that fewer than 1% of all patents are likely to be worth even 15 times the value of the mean patent. This finding underscores the difficulty that patentees face in obtaining full value for global patent portfolios, when the value in litigation of any individual patent's contribution to a complex product is likely to be small.

Keywords: intellectual property; patent; portfolio; valuation; litigation; profit; apportionment; statistics; evidence; royalty; damages.

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And that's the news from Lake Wobegone, where are all the women are strong, all the men are good-looking, and all the children are above average.

Garrison Keillor
A Prairie Home Companion[1]

1 Introduction

Almost by definition, the economic profit of a technologically complex product is likely to derive from multiple inputs, some of which may earn implicit economic rents. Often, these rent-earning inputs take the form of inventions protected by intellectual property rights. For example, a microprocessor may embody hundreds of patented inventions, in its circuitry, its manner of production and its packaging; similarly, an Apple iPhone is said to embody hundreds of patents (Gilroy and D'Amato 2009). Often, these inputs are owned and/or licensed for use by the manufacturer. Because ownership of a patent lacks the affirmative rights that economists normally associate with the ownership of physical inputs,[2] the web of supply contracts for patent rights (i.e., licenses) that are necessary to produce a technologically complex product is likely to be even more complicated than is suggested by the number of inputs alone.

When competitive relationships are relatively stable, competitors often contract around this complexity by offering blanket cross-licenses to each other's portfolios of patents, sometimes accompanied by a payment from the net importer to the net exporter of patent rights, on terms that are periodically renegotiated. Having given and received the "freedom to operate," firms then compete in other dimensions in the product market. As these competitive relationships become increasingly asymmetric, however, the likelihood of repeat negotiation diminishes, while the like-

[1] http://prairiehome.publicradio.org/about/podcast/. Though logically impossible (and therefore amusing), the "Lake Wobegone effect" is a recognized cognitive bias more formally known as "illusory superiority" (Van Yperen and Buunk 1991), due to which subjects overestimate their abilities relative to a peer baseline, such as the mean of a sample or population (*see* "Illusory Superiority," http://en.wikipedia.org/wiki/Illusory_superiority for a more complete discussion). For example, the vast majority of drivers state that they are better drivers than the median driver (Svenson 1981). Similarly, in a survey, 68% of the faculty at the University of Nebraska rated themselves in the top 25% in teaching ability (Cross 1977). Neale and Bazerman (1985) find that trial lawyers systematically overstate the likelihood that they will win at trial.

[2] Unlike the ownership of physical inputs, the ownership of a patented invention does not convey the right to use the invention, but only the right to exclude others from using it. For this reason, the owner of a patent (a) may not be allowed to use it (because owners of patents that are inputs to his invention may exclude his use) and (b) may derive value from it even when he does not use it (because he can prevent competing uses by others).

lihood of litigation increases. In the limit, the supplier of a technologically complex product may face claims from many owners of individual patents. When such litigation occurs, the question then arises: how much of the value of the complex product did any individual patent cause?

One such recent US litigation is *Lucent v. Microsoft*.[3] A jury awarded Lucent damages of about $358 million for Microsoft's infringement of a patent on a drop-down menu for selecting a date (the so-called "date-picker" feature), used in Microsoft's Outlook personal organizer software. Lucent's expert testified that Microsoft should pay a royalty of about $640 million (8% of about $8 billion in sales) were appropriate; Microsoft's expert testified that the payment should be $6.5 million, or about 1% of Lucent's demand.

In reviewing the evidence on appeal, the US Court of Appeals for the Federal Circuit (the "Federal Circuit," which hears most US patent appeals) found that, "The evidence can support only a finding that the infringing feature contained in Microsoft Outlook is but a tiny feature of one part of a much larger software program."[4] *Lucent* exemplifies a routine phenomenon: opposing economic experts differing by two orders of magnitude in their assessment of the share of profit attributable to an individual invention, particularly when the invention represents one input into a complex device generating sales in the billions of dollars.

The lack of even theoretical bounds on the share of value attributable to individual patents sharply increases the general unpredictability of litigation.[5] Insofar as litigation is the primary price discovery mechanism between the owner and user of an idiosyncratic, thinly traded asset, this uncertainty increases the costs of that discovery and reduces the likelihood of voluntary trade.

In an effort to rein in large, unsubstantiated claims, particularly those made by owners

[3] *Lucent Techs., Inc. v. Gateway et al.*, 580 F.3d 1301 (Fed. Cir. 2009). Microsoft appealed the district court judgment.

[4] Slip op., p. 48. The court went on to observe:

> We find it inconceivable to conclude, based on the present record, that the use of one small feature, the date-picker, constitutes a substantial portion of the value of Outlook.. . .
>
> . . . the only reasonable conclusion is that most of the realizable profit must be credited to non-patented [by Lucent] elements, such as "the manufacturing process, business risks, or significant features or improvements added [by Microsoft]."

Id., p. 49.

[5] The potential for biased (and therefore inefficient) pricing of patents in litigation has helped lead some economists to conclude that the patent system is simply "broken." *See* Jaffe and Lerner (2006) and Bessen and Meurer (2009).

of individual patents who sue manufacturers of complex devices, the Federal Circuit has adopted increasingly strict versions of the so-called "entire market value rule," which "allows for the recovery of damages based on the value of an entire apparatus containing several features, when the feature patented constitutes the basis for customer demand."[6] In *Uniloc v. Microsoft,* the Federal Circuit held that patentee may not even mention the level of the product sales to the jury, because bias inevitably ensues.[7] Needless to say, forbidding the use of product revenue as a metering device or other analytical input (except in the rare instance when the product is not technologically complex, and the invention can be deemed "the basis of customer demand") deprives analysts of one of the most essential and fundamental economic constructs. It is, moreover, an effort to restrict the use of analytically improper *methods* by instead restricting the analyst's use of *data*, and is likely to lead to further mispricing of patents, contrary to the Federal Circuit's express intent.

The risks of mispricing individual inputs can have outsize incentive effects on R&D, production and export decisions, potentially undermining or even reversing the incentive effects of the patent system itself.[8] This potential runs especially high in complex, highly integrated, products, where the infringement of even a handful of patents—each alleged to command a price far "above average"—can claim a disproportionate share of profit. To the extent that litigation-influenced prices are biased upward from true prices, that premium may attract an inefficiently large number of plaintiffs seeking excess returns that are available in the courtroom, but not in the market, thus fueling a litigation "explosion." Current efforts[9] to limit the litigation activities of patent assertion entities—sometimes known as "patent trolls"—stem in part from the allegedly disproportionate rewards that such litigants have been able to obtain through "abusive" litigation (though the Gov-

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[6](*TWM Mfg. Co. v. Dura Corp.*, 789 F.2d 895, 901 (Fed. Cir. 1986).

[7]

> This case provides a good example of the danger of admitting consideration of the entire market value of the accused [product] where the patented component does not create the basis for customer demand. As the district court aptly noted, "[t]he $19 billion cat [the revenue of the products accused of infringement] was never put back into the bag ... in spite of a final instruction that the jury may not award damages based on Microsoft's entire revenue from all the accused products in the case." This is unsurprising. The disclosure that a company has made $19 billion dollars in revenue from an infringing product cannot help but skew the damages horizon for the jury, regardless of the contribution of the patented component to this revenue.

632 F.3d 1292 (Fed. Cir. 2011).

[8]For example, Bessen and Meurer (2009) argues that the costs of (mostly defensive) patent litigation are equal in magnitude to the returns to patent protection obtained by large US firms.

[9]*See* "Fact Sheet: White House Task Force on High-Tech Patent Issues," http://www.whitehouse.gov/the-press-office/2013/06/04/fact-sheet-white-house-task-force-high-tech-patent-issues; "House Judiciary Chairman Goodlatte Seeks to Curb 'Patent Trolling'," http://www.usnews.com/news/articles/2013/10/24/house-judiciary-chairman-goodlatte-seeks-to-curb-patent-trolling.

ernment Accountability Office found that most of the recent historical increase in litigation was due to the rise of software-related patents, regardless of entity type; GAO 2013). But these procedural efforts skirt the central economic question: are the patents worth their asking price?

Bias-induced failures to reach agreement also confront infringers with the possibility of an injunction, which prevents the accused infringer from earning a return on the R&D, and potentially the hundreds of non-infringing inventions, that are also integrated into the infringer's complex product. In the standard-setting context, the question of whether a patentee has offered his patent on "fair, reasonable and non-discriminatory" (FRAND) terms, and whether it is in the public interest to permit the patentee to exclude the infringer's product, has assumed a central role in recent litigation,[10] particularly before the US International Trade Commission, which can prevent imports of infringing devices but lacks the statutory authority to set the price to be paid for either past or future use of the patent.

While asymmetric litigation by so-called "patent trolls" has recently assumed popular importance, it is by no means the only, or even the most economically meaningful, context in which uncertainty over the pricing of patents in technologically complex products may arise. Recent litigation between Oracle and Google,[11] and between Apple and Samsung[12] —both of which revolved around the assertion of a small number of patents that were alleged to have very high values—typify the difficulties that sophisticated repeat players face, not only in pricing and paying for intellectual property, but in the impact of their own and their rivals' enforcement efforts on their R&D and product design choices, which in turn help determine consumer choice and productivity growth.

In short, the need for accurate pricing of patents in non-market transactions goes beyond improved outcomes for private litigants to the efficacy of the patent system as a whole.

US patent law has long required that the price place on the use of an invention take account

[10] *See* the letter from Michael Froman, US Trade Representative, to Irving Williamson, Chairman of the US International Trade Commission (August 3, 2013), disapproving an exclusion order issued by the ITC against certain Samsung products for infringing Apple patents (available at: http://www.ustr.gov/sites/default/files/08032013%20Letter_1.PDF); US Department of Justice / US Patent and Trademark Office, *Policy Statement on Remedies for Standards-Essential Patents Subject to Voluntary F/RAND Commitments,* January 8, 2013 (available at: http://www.uspto.gov/about/offices/ogc/Final_DOJ-PTO_Policy_Statement_on_FRAND_SEPs_1-8-13.pdf).

[11] http://www.groklaw.net/staticpages/index.php?page=OracleGoogle

[12] http://www.groklaw.net/staticpages/index.php?page=AppleSamsung

of the portion of the infringer's profit that his invention has caused[13] and, if that portion is less than the whole, to apportion the infringer's profit between that which is legally attributable to the use of the patentee's invention and that which is attributable to the infringer's own inputs.[14]

Like many other aspects of damages calculations, the apportionment of profit has long been misstated, manipulated, ignored and otherwise abused by litigants and their witnesses. Apart from misconduct, there are multiple reasons for this state of affairs:

- "profit" is not well-defined in patent damages law, and is generally taken to mean accounting profit earned over the interval of infringement[15]
- even if profit were well-defined, the allocation of that profit to discrete causes may be difficult to determine, both conceptually and empirically
- even assuming the accurate definition and allocation of profit, the law looks to "realizable" profit (*i.e.*, actual or forecast prior to the determination of liability), not to the profit that

[13]35 U.S.C. §284 awards damages "adequate to compensate for the infringement, but in no event less than a reasonable royalty for the use made of the invention." Because damages are compensatory, disgorgement of the infringer's profit is not a remedy available to the patent owner (as it may be for, say, copyright or trade secret owners), so the infringer's profit (or a portion thereof) is not a direct measure of damages. Further, if the patentee can show that the infringement has caused the patentee to lose sales that he otherwise would have made, the patentee's (entire) profit on those lost sales often may be claimed as damages, generally without further apportionment. When the patentee cannot show such a loss, the alternative measure of damages is a reasonable royalty on the infringer's sales. It is in the context of setting the royalty on the infringing sales that distinguishing among the source(s) of the infringer's profit is relevant. This paper assumes that context.

[14]Among the so-called "*Georgia-Pacific* factors," which frame the "hypothetical negotiation" that is supposed to yield a reasonable royalty for the infringing use, factor 13 requires the trial court to examine "the portion of the realizable profit that should be credited to the invention, as distinguished from non-patented elements, the manufacturing process, business risks, or significant features or improvements added by the infringer." *Georgia-Pacific Corp. v. United States Plywood Corp.*, 318 F. Supp. 1116, 1119-20 (S.D.N.Y. 1970), *modified and aff'd*, 446 F.2d 295 (2d Cir.).

See also Lucent, tracing the history of profit apportionment to *Garretson v. Clark* (1884), which held:

> When a patent is for an improvement, and not for an entirely new machine or contrivance, the patentee must show in what particulars his improvement has added to the usefulness of the machine or contrivance. He must separate its results distinctly from those of the other parts, so that the benefits derived from it may be distinctly seen and appreciated. . . . The patentee . . . must in every case give evidence tending to separate or apportion the defendant's profits and the patentee's damages between the patented feature and the unpatented features, and such evidence must be reliable and tangible, and not conjectural or speculative; or he must show, by equally reliable and satisfactory evidence, that the profits and damages are to be calculated on the whole machine, for the reason that the entire value of the whole machine, as a marketable article, is properly and legally attributable to the patented feature.

[15]Perhaps most importantly, current accounting profit may not take into account returns to past R&D (performed by the infringer prior to the damages period). See Fisher and McGowan (1983); for an alternative view, see Martin (1988).

would have occurred had the defendant properly licensed the patent and priced his product accordingly

- even if these other requirements were met, courts typically do not require proof of the analytical corollaries of a standard apportionment—such as that the apportioned "parts" add up to a relevant "whole"

This paper focuses on the last of these deficiencies. Drawing on the economic literature on the distribution of patent values, I describe simple procedures for computing the share of a given sample that can reasonably be attributed to any one patent, assuming the presence of an adding-up constraint. I express these shares relative to the mean of the distribution, which is—unlike profit shares themselves—often easy to observe or bound. While subject to further refinement in light of the facts of individual cases, these procedures improve the inferences to be drawn from the data and may set bounds on the claims that litigants can make regarding the contribution of an individual invention to the profit of a complex product. More generally, these methods can be used to extract the market value of a single patent to be inferred from transactions involving a larger whole, as when firms license or sell large groups of patents.

Section 2 lays out the assumptions and notation of the paper. Section 3 reviews the empirical literature on the distribution of private patent values. Section 4 shows how to employ this literature to construct shares of value based on rank. Section 5 gives some applications, and Section 6 offers some concluding remarks.

2 Notation, Assumptions and Applications

2.1 Notation and assumptions

We begin with a technologically complex product that derives value[16] from a portfolio of P patents, which we take to be drawn independently from a common distribution. The aggregate value of the

[16] For certain legal purposes, a product must actually practice the claims of patent. More generally, a product may derive value from a patent because the patent inhibits competing entry, even if the product does not itself practice the patent's claims. For present purposes, the distinction is irrelevant.

portfolio is V, which could be the product's profit or more generally the price required to license the portfolio. The value of each patent i is v_i. The general problem we face is how to determine the share s_i of V that is attributable to each patent. I make the following assumptions throughout the paper:

$$v_i > 0 \quad \text{(non-negativity)} \tag{1}$$

$$\sum_{i=1}^{P} v_i = V \quad \text{(adding-up)} \tag{2}$$

$$s_i = v_i / V \quad \text{(value shares)} \tag{3}$$

$$s_{i+j} = s_i + s_j \text{ for all } i \neq j \quad \text{(linear aggregation)} \tag{4}$$

In other words: the value of a share is greater than zero; when combined, two shares must equal their sum; and the whole must equal the sum of the parts.

In particular, I interpret "apportionment of value" to require satisfaction of the value adding-up constraint (2). A valuation that conforms to (1)-(4) is said to be "AUC-consistent."

It is sometimes useful to express the unconditional mean value $\overline{v}$ as $\overline{v} = \overline{s}\, V$, where $\overline{s} = 1/P$ is the mean patent share.

Without loss of generality, we assume that we can order the patents in the portfolio from least to greatest share. Thus s_n is the share of the patent occupying the n^{th} percentile of the distribution, $0 < n \leq 1$, with $s_n \geq s_{n-x}$ for $0 < x < n$. This ordering is assumed throughout.

It is often useful to compute partial sums. For the n^{th} percentile patent, $0 \leq n < 1$, let

$$V_n = \sum_{i=1}^{nP} v_i \tag{5}$$

be the sum of the bottom nP patents. Then V_n/V is the share of total value attributable to these patents. We define

$$L_n = \frac{(V_n/V)}{nP/P},\ 0 < L_n \leq 1 \tag{6}$$
$$= \frac{V_n/nP}{V/P}$$
$$= \overline{v}_n/\overline{v}$$

as the ratio of the mean of the lowest $n\%$ of patents to the overall mean. A Lorenz graph plots the numerator of (6) against the denominator.

By virtue of (1-4), shares can be also be defined for an arbitrary collection of patents. It is frequently useful to compute the aggregate share of patents over a continuous range. Let u be the upper limit of a range, and let V_u be the sum of the bottom uP patent values,

$$V_u = \sum_{i=1}^{uP} v_i,\ v_i \geq v_{i-x},\ 0 < x < u$$

and similarly for V_l. Then $s_l^u = (V_u - V_l)/V$ is the share of total value attributable to patents in the range (l, u). By definition, these patents represent $(u - l)\%$ of the total. It follows that the conditional mean of a patent over the range (l, u) is

$$\overline{v}_l^u = \frac{V_u - V_l}{P(u-l)} = \frac{\Delta V}{\Delta P}$$

or in percentage terms,

$$\Delta L_l^u = \frac{(V_u - V_l)/V}{P(u-l)/P} = \frac{\overline{v}_l^u}{\overline{v}} = \frac{\overline{s}_l^u}{\overline{s}} \tag{7}$$

Given the extreme skew of the distribution of v, and the focus on allegedly high-value patents in litigation, it is sometimes useful to examine a special case of (7): the upper tail of the value distribution. Define

$$M_n = \frac{1 - V_n/V}{1-n} = \frac{(V - V_n)/(P - nP)}{V/P} = \frac{\overline{v}_n^{\max}}{\overline{v}} = \frac{\overline{s}_n^{\max}}{\overline{s}} \geq 1 \tag{8}$$

to be the ratio of the conditional mean of the top $(1 - n)\%$ of the distribution to the unconditional mean.

When precise information about a patent's rank is available, that precision can be used to state the exact relationship between the value of a patent occupying the n^{th} percentile and the average patent. For $l < n < u$, define

$$K_n = \lim_{(u-l)\to 0} = \frac{\Delta V/\Delta P}{V/P} = \lim_{(u-l)\to 0} \frac{\overline{v}_l^u/V}{1/P} = \frac{v_n/V}{1/P} = \frac{v_n}{\overline{v}} = \frac{s_n}{\overline{s}} \tag{9}$$

Note that K_n is the slope of a Lorenz graph evaluated at the point n.

Finally,

$$\sum_{i=1}^{P} K_i = \sum_{i=1}^{P} \frac{v_i}{V/P} = \frac{V}{V/P} = P \tag{10}$$

That is, in a AUC-consistent apportionment, the sum of the ratios K_n for all patents in the portfolio must equal the number of patents in the portfolio.

2.2 Litigation

Apportionment problems arise when a price (or a non-market valuation) is observed at a level of aggregation, such as a patent portfolio, greater than an individual patent[17] or some smaller aggregate that must be priced. For example, a patent portfolio generates an aggregate royalty payment R. Then $\overline{r} = R/P$ is the average royalty per patent, and $r_n = s_n R$ is the royalty attributable to the n^{th} patent. Similar issues may arise when patents are sold as a group, then

[17] In principle, the same analysis applies to other types of intellectual property, such as copyrights and trade secrets. These other types suffer from various additional difficulties that make their apportionment more difficult:

1. P. Other types of IP are often maintained as cumulative, rather than discrete, legal assets. For example, within Microsoft Office appears the following notation: "Import/Export Converters ©1988-1998 DataViz, Inc." Original improvements to the Dataviz software code made over the decade 1988-1998—all of which are generally copyrightable—likely became part of a single underlying copyright, rather than being protected (and countable) separately.
2. V. Other types of IP are less frequently traded as a group (for example, one rarely observes a "copyright portfolio cross-license"). One does sometimes observe the pricing of a package of trade secrets or other know-how in technology transfer agreements, but this is generally defined functionally ("all information necessary to accomplish x"), rather than in terms of constituent discrete facts.
3. s_n. Most other types of IP are not subject to observable optimizing behavior, such as the payment of application or renewal fees. This is a necessary condition for estimating a structural model, like those reported for patents in Table 1. Structural models define the distribution of asset values, from which to compute the share attributable to each constituent asset.

Without further assumptions, these difficulties generally leave P, V and s_n difficult to calculate, or even define.

subdivided, or when the sources of aggregate profit must be allocated among individual causes. Alternatively, there are various reasons to extrapolate from a known part to some larger, but unobservable, whole.[18]

In litigation—which focuses almost entirely on the asserted patent, with little or no mention of others in the portfolio—patentees routinely claim that v_n is "large," while accused infringers claim that it is "small." Through some combination of (1)–(4), these claims can often be expressed in terms of P, V, $\overline{v}$ (or their royalty, asset and profit analogues, as applicable) and s_n, and tested for their internal consistency and/or consistency with other data. It will be convenient to assume that P, V and $\overline{v}$ and/or their analogues are observable or otherwise not disputed. This assumption allows us to focus on the determination of s_n—*i.e.,* on apportionment. Section 4 provides values of K_n, M_n and ΔL_l^u that make these determinations easy to implement from the empirical literature.

3 Empirical Literature

Beginning with Pakes and Schankerman (1984), economists have employed a variety of methods to derive the distribution of patent values from optimizing behavior. Broadly speaking, these efforts can be divided into two types: "longitudinal" models of patent renewal decisions, and "cross-sectional" models of patent family (country choice) decisions. Each of these types can be further divided into "perfect foresight" models, which assume that initial returns decay at a deterministic rate, and "option" models, which permit returns to evolve stochastically. Analysts typical assume that initial returns are distributed log-normally.[19]

Papers based on patent renewal methods generally provide estimates for individual European countries, because there are too few renewal decisions during the life of a US patent to identify the distribution. Lanjouw et al. (1998) surveys most of the relevant research. Since then,

[18]For example, Teece (2000, p. 207) describes a procedure by which prospective cross-licensees each create a list of their top patents (a "proud list"), and rate them in various dimensions: likelihood of infringement, validity and next-best alternatives. Each patent receives an aggregate score, which is then multiplied by a common royalty rate. The sum of these weighted royalties constitutes each firm's aggregate claim on the other. According to Teece, the complete analysis of a pre-2000 complex semiconductor device could require 400-500 hours.

[19]The log-normal distribution can be justified theoretically by modeling technical change as the product of independent multiplicative improvements to a production function, and appealing to a central limit theorem. Schankerman and Pakes (1986) reported that the log-normal distribution fit best among the several distributions they tested.

Bessen (2008) estimates a deterministic model based on US patent renewals, using observed patent covariates (such as subsequent citations in later patents) to identify the value distribution. Among papers analyzing the choice of international patent family, Deng (2011) has integrated the various strands of the literature into a single general model of (European) patent family and renewal decisions with stochastic returns. Chan (2010) estimates an international patent family application model using firm-level data for the agricultural biotechnology sector.

All of these models are identified, partially or completely, using the behavioral assumption that each year a patentee compares the annual return to patent protection, r_t, to the annual cost of maintaining that protection, c_t, and responds optimally by either renewing the patent, or not. In each model, the distribution estimated is that of the initial annual return, r_1 (which captures the first-year return received by the inventor).[20] Patent family models assume, in addition, that in each country j in which an application is observed the capitalized asset value (over the life of the patent) exceeds the initial application cost: $v_j = \sum_{t=1}^{T^*} \beta^t (r_{jt} - c_{jt}) > C_{j0}$, where C_{j0} is the cost of filing in country j, $T_j^* \leq \overline{T}_j$ is the length of the patent's life (assuming optimal renewal decisions, subject to the statutory maximum life $\overline{T}_j$), $0 < \beta < 1$ is a discount factor, and r_{j1} depends in addition on an invention-specific random effect that is common across countries.

For present purposes, the salient conclusions of these papers are:

1. The assumption that r_1 is distributed log-normally fits the data well (and better than other parametric distributions).

2. Though the studies naturally reach different conclusions on the mean value of patent rights,[21] (which varies with the log-normal location parameter μ_{r_1}) the scale parameter σ_{r_1} falls within a relatively narrow range (typically about $1.5 - 2.2$).

To facilitate the exposition, I assume log-normality of the value distribution (*i.e.*, that $\ln v \sim N(\mu_v, \sigma_v)$),[22] but as Section 4.3 shows, this assumption is not essential to the main results

[20] In "deterministic" models, r_t is typically modeled as $r_t = r_1 \delta^{t-1}$, where $\ln r_1 \sim N(\mu_{r_1}, \sigma_{r_1})$ and $0 < \delta < 1$ is a depreciation factor. In "stochastic" models, r_t evolves according to a stochastic process which permits the value of the patent to increase, though with decreasing probability, over time. The two methods generally imply similar valuations in the right tail of the distribution, *i.e.,* among high-value patents.

[21] For example, patents are worth less in smaller countries, and values vary systematically by technology field.

[22] The distribution of v is approximately, but not exactly, log-normal, because (a) v is the sum of log-normals,

of the paper.

Almost all of the studies report v for the samples they examine at standard percentiles (generally at least 0.25, 0.50, 0.75, 0.90, 0.95 and 0.99). From the reported values, it is simple to compute the implied distribution of v over the range between two percentiles, assuming log-normality. Let v_u be the value of v reported for the upper percentile, and v_l be the value reported for the lower percentile. Then we have $v_u = \exp[\mu_v + \sigma_v F^{-1}(u)]$ and $v_l = \exp[\mu_v + \sigma_v F^{-1}(l)]$, where $F(\cdot)$ is the standard normal distribution function. Solving these equations jointly for μ_v and σ_v gives:

$$\sigma_v = (\ln v_u - \ln v_l)/[F^{-1}(u) - F^{-1}(l)] \tag{11}$$

Table 1 reports the values of σ_v inferred using (11) from the estimates reported in various studies. These studies examine the patent value distribution across a range of countries, technologies and time periods, using both patent renewal and patent family application methods. With the exception of Bessen (2008) and Chan (2010), all of them report the simulated distribution of patent values at the same percentiles.[23]

Table 1: Value of σ_v implied from various patent renewal and patent application studies

Estimates of σ_v vary somewhat over the different intervals between percentiles. In general, the implied value of σ_v decreases in the right tail of the distribution, partly because truncation has less effect on the values there.

The table highlights several differences among the studies, and hints at possible explanations for those differences. For example, unlike other studies, Lanjouw (1998) assumes an exponential

which is not log-normal; (b) the application fee C_0 and annual renewal fees c_t cause inventors who draw a low value for r_1 either not to file at all, or to allow their patents to lapse prior to reaching $\overline{T}$; both of these distortions truncate the left tail of the distribution of v; and (c) these fees constitute a larger proportion of lower-value patents, further skewing the v distribution (which, unlike the distribution of r_1, is based on value net of fees). For these reasons, the implied values of σ_v vary somewhat over the support of v. Note that any sample drawn from the log-normal distribution satisfies assumptions (1)–(4), and is therefore AUC-consistent, provided that the draws are independent.

[23] The value reproduced from Bessen (2008) is the middle of three estimates reported for σ_{r_1}, which generally falls near the middle of the implied range for σ_v. Chan (2010) reports percentiles corresponding to points in the support of the distribution that are fixed across the countries she studies. I calculate σ_v for the percentiles that most nearly match those shown in Table 1.

distribution of initial returns, which (having a thinner right tail) appears to yield lower estimates of σ_v than are found by authors who assume a log-normal distribution. In general, the international patent family studies find higher estimates for σ_v than those found in studies based on patent renewal data.[24] On the other hand, there appears to be no systematic difference between studies that assume perfect foresight and those that allow for the stochastic evolution of returns.

For present purposes, the main conclusion from Table 1 is that the values of σ_v fall within a relatively narrow range.[25] The median value of σ_v ranges from about 2.1 for the bottom 75% of the distribution to about 1.7 for the top 5%.

It is important to study the impact of variations in σ_v on the value distribution, to understand the circumstances in which such variation does and does not matter for the apportionment of value. Figure 1 plots Lorenz graphs (from (6)) for values of σ_v ranging from 0 to 3. A value of 0 implies that all patents have the same value (the "45 degree line"). As σ_v increases, the distribution becomes increasingly skewed, with the right tail of the distribution commanding an increasing fraction of the total value. For example, for $\sigma_v = 1$, the bottom 90% of the distribution represents about 61% of the total value ($L_{90} = 0.68$), which implies that the top 10% of patents constitute 39% of the total ($M_{90} = 3.9$). For $\sigma_v = 2$, the figures are: $L_{90} = 0.24$, $M_{90} = 7.6$.

Figure 1: Lorenz graphs of the patent value distribution for selected σ_v

The heavy line in Figure 1 plots the composite Lorenz graph constructed using the medians of the estimates reported in Table 1. The median graph lies between those plotted for $\sigma_v = 1.5$ and $\sigma_v = 2$.

Though the aggregate skewness of the distribution varies markedly, depending on σ_v, the next section shows that the shares of the distribution are quite stable over the vast majority of the

[24] Chan (2010), a study of agricultural biotechnology inventions that were made by commercially successful firms, consistently shows the highest estimates across all studies. Such patents may exhibit greater skew.

[25] Relatively small variations in σ_v, by themselves, imply relatively large variations in the expected value of the underlying distribution, which is given by $E[v] = \exp(\mu_v + \sigma_v^2/2)$. But we are interested in the *shares* of the distribution, and take the expected value as having been estimated from other data (such as average profit per patent). Variations in σ_v are relevant only insofar as they affect the implied shares. As I explain, shares do not vary much with σ_v, except in the extreme right tail.

distribution.

4 Value Shares

4.1 Arc and point estimates

Given a portfolio of P patents valued at V, the exact share s of V for any patent (or range of patents) is

$$\begin{aligned} s &= V^{-1}\ P \int_L^U dG(v)\ dv \int_L^U v\ dG(v)\ dv \\ &= V^{-1}\ P\ [G(U) - G(L)]\ \overline{v}_L^U \\ &= [u - l]\ \frac{\overline{v}_l^{\,u}}{\overline{v}} \end{aligned} \tag{12}$$

where G is the patent value distribution function and L and U are the lower and upper limits of the range, respectively. Assuming that G is log-normal with parameters (μ_v, σ_v), the share s_n is accurately approximated for large P by

$$s_n \approx V^{-1}\ \exp[\mu_v + \sigma_v\ F^{-1}(n)] \tag{13}$$

where F is the standard normal distribution function and n is the percentile at which the share is evaluated.[26] Since the expected value of a log-normal distribution is $E[v] = \exp(\mu_v + \sigma_v^2/2)$, the expected value of all patents in the portfolio is

$$V = P\ \exp(\mu_v + \sigma_v^2/2) \tag{14}$$

We can use (14) to rewrite (13) as:

$$s_n \approx P^{-1} \exp[-(\mu_v + \sigma_v^2/2)]\ \exp[\mu_v + \sigma_v\ F^{-1}(n)] \tag{15}$$

[26]The approximation is not exact because (12) is a range and (13) is evaluated at a point. For large P, the integral of the range corresponding to a single patent occupying the n^{th} percentile (12) is very close to the value of (13) evaluated at the point n.

It is easy to see that s_n depends only on σ_v, since (15) can be written as:

$$\begin{aligned} s_n &\approx P^{-1} \exp(-\mu_v)\ \exp(\mu_v)\ \exp(-\sigma_v^2/2)\ \exp[\sigma_v\ F^{-1}(n)] \\ &= P^{-1} \exp[\sigma_v(F^{-1}(n) - \sigma_v/2)] \end{aligned} \tag{16}$$

It is also easy to show that s_n reaches a maximum at $\sigma_v^* = F^{-1}(n)$.

Because σ_v varies over the value distribution, one cannot select a single log-normally derived Lorenz graph to characterize the distribution. As it turns out, however, this limitation does not matter much for the purpose of computing value shares, at least for most patents. Since shares vary with the slope of the Lorenz graph, Figure 1 shows that this slope does not vary much with σ_v for patents below about the 90^{th} percentile. In other words, for the vast majority of patents, it makes very little difference what value one assigns to σ_v, because the implied shares change very little, and are small in any event.[27]

Even above the 90^{th} percentile, shares are relatively stable across a broad range of σ_v. Figure 2 makes this observation more precise, by plotting s_n for various values of σ_v in a portfolio of $P = 1000$ patents. For example, over the range reported in Table 1 ($\sigma_v \in [1.1, 3.2]$), s_{90} decreases from about 0.22% to 0.04% of total value, and equals about 0.20% when evaluated at $\sigma_v = 1.71$, the median reported value.[28] By comparison, over the reported range $\sigma_v \in [0.8, 2.8]$, s_{99} ranges from about 0.5% to about 1.5%, and equals a little more than 1.2% when evaluated at the median.[29] Of course, the reported shares are inversely proportional to P as long as P remains "large." For example, for a portfolio with half as many patents ($P = 500$), s_{99} is twice as large, ranging from about 1% to about 3%.[30]

Figure 2: Shares s_n of total patent portfolio value, for selected σ_v and n

[27]Patents below the 75^{th} percentile have shares below the mean for all reported values of σ_v.

[28]The dotted lines in Figure 2 show the limits of the range of values reported in Table 1.

[29]Because (16) achieves a maximum at $\sigma_v^* = F^{-1}(n)$, s_n is not monotone: as the distribution becomes increasingly skewed (*i.e.*, as σ_v increases), draws from the extreme tail of the distribution command larger shares, only to see those shares shrink as even more extreme draws are assigned even greater shares for higher values of σ_v. For example, the share of the 98^{th}-percentile patent s_{98} peaks at $\sigma_v^* = F^{-1}(0.98) = 2.05$. For $\sigma_v > 2.05$, s_{98} is decreasing, but the share of the 99^{th}-percentile patent s_{99} is increasing, up to $\sigma_v^* = F^{-1}(0.99) = 2.33$, where s_{99} also begins to decline.

[30]These extrapolations become less accurate as P becomes "small," and for extreme values of the distribution. Of course, equation (12) still gives an exact share for any P and n, $l < n < u$.

A simple way to apply these relationships to portfolios of other sizes is to express shares in terms of their ratio to the mean share, $K_n = s_n/\overline{s}$ (from (9)), as shown by the right-hand vertical axis in Figure 2, in which the mean share is $\overline{s} = 1/P = 0.1\%$. Figure 2 shows that (assuming $\sigma_v = 2.0$), an AUC-consistent estimate of the share attributable to the 99^{th}-percentile patent in a 1000-patent portfolio is $s_{99} = 1.42\%$, while $K_{99} = 1.42\%/0.1\%$, or about 14.2. In other words, regardless of portfolio size, the share of a 99^{th}-percentile patent is about 14.2 times the share of the average patent.

In general, since $K_n = s_n/\overline{s}$, we can use (16) to compute K_n for any pair (n, σ_v):

$$K_n = \exp[\sigma_v(F^{-1}(n) - \sigma_v/2)] \tag{17}$$

Because

$$K_n = s_n/\overline{s} = (s_nV)/(\overline{s}V) = v_n/\overline{v} \tag{18}$$

the ratio K_n can be used to directly infer the value of the n^{th} patent, if one knows the mean value of all patents $\overline{v}$: $v_n = K_n\ \overline{v}$. Moreover, unlike s_n itself, K_n is essentially invariant to the number of patents in the portfolio, as long as P remains "large."

Table 2 reports the values of K_n for representative pairs of σ_v and n, as well as for the median values of σ_v reported in Table 1. One can readily see that, except for patents ranked at about the 97^{th} percentile or higher, these ratios are relatively stable across a broad range of assumed values for σ_v. And, even for the highest percentiles, the reported values still provide a reasonably narrow range of possible valuations. In particular, they offer a helpful check on improbable claims.

Table 2: K_n: the ratio of the n^{th} percentile patent value to the mean patent overall

When a patent's ranking can be determined only approximately, it may be helpful to work with averages over a range of the value distribution (l, u), as in (7). Under such circumstances, one assumes that the patent in question has an average conditional value, *i.e.,* conditional on falling within the indicated range.

Table 3 shows the ratios ΔL_l^u for various combinations of l and u, using the median Lorenz graph from Figure 1. For example, the average patent that falls in the percentile range (90, 95) is worth about 2.7 times the average patent overall.

Table 3: K_l^u: the ratio of the mean patent in (l, u) to the mean patent overall

More particularly, suppose evidence suggests that a patent is drawn from the top $(1 - n)\%$ of the distribution, but further precision is difficult or expensive to obtain.[31] One can then compute its ratio to the average patent, $v_n = M_n\ \overline{v}$ (from equation (8)). Table 4 provides values of M_n for various combinations of n and σ_v. For example, using the median of the reported values for σ_v (the last column of Table 4), the average of all patents above the 95^{th} percentile is 10.2 $\overline{v}$, or about 10 times the value of the average patent. Note that this estimator of v_{95} is considerably larger than that obtained from Table 2 (3.7 $\overline{v}$), which would be appropriate if it were known that the patent occupied exactly the 95^{th} percentile.

Table 4: M_n: The ratio of the mean value of patents above the n^{th} percentile to the mean patent overall

Tables 3 and 4 illustrate a relationship that proves useful when testing valuation claims. Note that every portfolio can be divided into two subsets: those ranked higher than n, and those ranked lower. From equation (12), the shares of total value corresponding to these subsets are given by $G(n)\ \overline{v}_0^{\,n}/\overline{v}$ and $[1 - G(n)]\ \overline{v}_n^{\,\max}/\overline{v}$. From (6), (8) and (2), this implies:

$$nL_n + (1 - n)M_n = 1 \tag{19}$$

For median values of σ_v, L_n can be read directly from the first row of Table 3, while M_n can be read directly from the last column of Table 3 (or, for other values of σ_v, from Table 4). Equation (19)

[31] In *LG Display Co., Ltd., v. AU Optronics Corp.*, No. 06-726 (JJF) (D. Del.), the court based its patent infringement damages award on the assumption that each of the infringed patents had a value equal to the average of the top 5% of all patents in the value distribution. *See* Memorandum Opinion, July 8, 2010, available at: http://www.scribd.com/doc/34218584/LG-Display-CO-Ltd-V-AU-Optronics-Corporation-et-al-C-A-No-06-726-JJF-D-Del-July-8-2010. The author testified on behalf of AU Optronics.

states that a weighted average of these statistics, with weights derived from the patent's percentile rank n, always equals 1. For example, for a patent ranked in the 95^{th} percentile, we have $0.95 \times 0.517\ +\ 0.05 \times 10.2\ = 1$.

Equation (19) restricts the values that can be claimed for a patent of rank n. In particular, a patent of rank n is, by definition, the least valuable among all of the patents that constitute the range $(n, 1)$. Therefore, its value v_n must be less than or equal to the average of all patents that rank above it, M_n (with the equality holding only if the n^{th} patent and all the patents that rank above it have identical values). Any greater claim for a patent of rank n is equivalent to arguing that all patents in the range $(n, 1)$ are above the average for the range, which is of course impossible.[32]

4.2 Monte Carlo confidence intervals

Because of the extreme skew of the patent distribution, and therefore the wide variability in small samples, any given patent portfolio need not conform exactly to the relationships reported above, especially as sample sizes decrease. To give an idea of the relationship between value shares and portfolio size in small samples, I drew 10,000 portfolios of various sizes and computed the ratio to the mean value (*i.e.,* the empirical K_n) for patents at the standard quantiles of the distribution. I assumed that $\sigma_v = 1.71$.

I calculated upper and lower confidence bounds for these values, expressed relative to the expected value of K_n (given by (17)). For example, $K^u_{0.75}$ is the upper bound of the 95% confidence interval for the 75^{th} percentile of the value distribution, expressed as a ratio to $E[K_{75}]$. Figure 3 displays confidence intervals for representative K_n. While there is some divergence for small portfolios, these intervals are very similar for portfolio sizes of $P = 100$ or greater. For example, for $P = 1000$, the confidence interval is approximately $\pm 20\%$, with a slightly larger upper limit for K_{99}.

[32] Similarly, a patent of rank n is, by definition, the most valuable among all of the patents that constitute the range $(0, n)$. Therefore, its value v_n must be greater than or equal to the average L_n of all patents that rank below it (with the equality holding only if the n^{th} patent and all the patents that rank below it have identical values).

Figure 3: 95% confidence intervals for $\hat{K}_n$ relative to the expected value, for selected P

4.3 Robustness

This section considers various modifications to the basic allocation model that may be proposed or required by circumstances, and the sensitivity of the model's results to those modifications.

Sample selection. It is natural to suppose that the patents that are practiced in a complex product represent a selection of the best of the patentee's inventions, rather than a random sample.[33] We implement this notion by assuming that the set P is a sample selected from a superset P^+ having total value V^+. To introduce the greatest selection bias possible, we assume that P represents the top $1-m$ patents in P^+. Thus, the observed data permit us to calculate $K_{n|m}$, where $n|m$ is the n^{th} percentile of the observed sample of P patents, $P = (1-m)P^+$. We seek expressions for $K^*_{n|m}$ and $s^*_{n|m}$, the corrected weights and shares.

First, note that $n = 1-(1-n|m)(1-m)$,[34] and that relative to the unconditional mean $\overline{v} = V^+/P^+$, the mean of the observed sample is $M_m\overline{v}$ (Table 4). It is straightforward to show that

$$K^*_{n|m} = \frac{K_n}{M_m} \tag{20}$$

where K_n is the weight calculated from the patent's true percentile.[35] Thus, the ratio $K^*_{n|m}/K_{n|m}$

[33]The mechanism by which inventions are selected into a complex product itself likely represents the interaction of many complex (and unobservable) factors: strategic and technical complementarity with other features and inventions; costs and risks of implementation; path dependence resulting from prior design decisions; rival product design and patenting; etc.

It should be emphasized that a product's profit may be caused in part by patents that the product itself does not practice. For example, suppose a product can be produced by one of three methods at marginal cost c, with $c_1 < c_2 < c_3$. By selecting method 1 but patenting method 2 as well, a firm reduces competitive pressure from rivals who could otherwise produce using method 2. Thus, at least from an economic point of view, the relevant set comprises not (only) the patents that the product "embodies," but the patents that confer competitive or other economic advantage.

[34]For example, we observe a sample of $P = 200$ patents, which is drawn from the top 20% of a sample of $P^+ = 1000$ patents (so $m = 0.80$). A patent that ranks 11^{th} in the observed sample ($n|m = 0.95$) implies a true percentile of $n = 0.99$ out of P^+.

[35]To continue the prior example: if no correction for sample selection is made, the 11^{th}-ranked patent in a sample of 200 has an expected value equal to $K_{95} = 3.86$ times the observed sample mean. Under the assumption that these 200 patents constitute the top 20% of a sample of 1000, the expected value should be $K_{99}/M_{80} = 12.38/4.03 = 3.07$ times the observed sample mean.

is the ratio of corrected to uncorrected weights. Similarly,

$$s_n^* = s_n \frac{K_{n|m}^*}{K_{n|m}}$$

Perhaps counterintuitively, $K_{n|m}^*/K_{n|m} < 1$ for the far right tail of the observed sample. This means that sample selection *reduces* the share of total value imputed to the most valuable patents.[36]

Figure 4 plots this ratio for various values of n and m. For $m = 0.80$, the ratio for the 11^{th}-ranked patent is 3.07 / 3.86 = 0.8. The greatest distortions from sample selection are found for the least valuable patents.

Figure 4: Ratio of $K_{n|m}^*$ to $K_{n|m}$ for selected n and m

Strategic misrepresentation of data. A potentially more troublesome source of bias, and an obvious litigation ploy by accused infringers, is to inflate the size of the "relevant" patent portfolio, P. Call the inflated claim $\tilde{P}$ and the true number P^*; similarly, the claimed and true percentiles are denoted $\tilde{n}$ and n^*, respectively. Then $\Delta P = \tilde{P} - P^*$ is the number of improperly added patents, and $c = \Delta P/P^*$ is the inflation factor.

Undoing the effects of inflation depends on the assumption one makes about how and where the irrelevant patents were added to the rankings. As a benchmark, suppose that the ΔP were added randomly throughout the rankings. Then, in expectation, the percentile rank does not change when these patents are removed (*i.e.*, $n^* = \tilde{n}$), but the mean share increases from $1/\tilde{P}$ to $1/P^*$. In that case, removing the improperly added patents increases the true value (or share) of the n^{th} patent by the factor c.[37]

[36] In other words, the right tail of the lognormal distribution is distributed more equally than the distribution as a whole.

[37] For example, suppose that $\tilde{P} = 1100$, and the defendant claims that $\tilde{n} = 0.90$, the 90^{th} percentile (*i.e.,* the asserted patent ranks 110^{th} in the portfolio). Suppose that the true portfolio has $P^* = 1000$ patents; thus $\Delta P = 100$ and $c = 0.10$. Suppose that the extra ΔP patents are located randomly in the sample. After these are removed, the patent is expected to rank 100^{th} out of 1000, *i.e.,* $n^* = \tilde{n} = 0.90$, as before. Since $K_{n^*} = K_{\tilde{n}} = 2.1$, and the mean share increases from 1/1100 to 1/1000, the share of the asserted patent increases by a factor of $1 + c = 1.1$, from $2.1/1100 = 0.19\%$ to $2.1/1000 = 0.21\%$.

In general—because the parties have obvious incentives to add or subtract patents ranked higher than n—there is likely to be disagreement as to the rank of the improperly added patents. Let d be the share of ΔP that rank below $\tilde{n}$, $0 \leq d \leq 1$; the benchmark case corresponds to $d = \tilde{n}$. Assume instead that the ΔP improper patents are divided into two subsamples of size $d\Delta P$ and $(1-d)\Delta P$, but that within each subsample the patents are distributed randomly.[38] One can easily show that the patent's true expected rank is:

$$n^* = \tilde{n} - c(d - \tilde{n}) \tag{21}$$

(subject to the constraint that $n^* \leq \tilde{n}(1+c)$). When the improperly added patents are removed, the increase in the patent's value (or share) is expected to be (using (17) and (21) and simplifying):

$$\begin{aligned} \frac{v^*}{\tilde{v}} = \frac{s^*}{\tilde{s}} &= (1+c)\ (K_{n^*}/K_{\tilde{n}}) \\ &= (1+c)\ \exp[\sigma_v(F^{-1}(\tilde{n} - c(d-\tilde{n})) - F^{-1}(\tilde{n}))] \end{aligned} \tag{22}$$

In short, a plaintiff can correct a defendant's improper valuation claim $\tilde{v}(\tilde{P}, \tilde{n})$ by proving (c, d) and multiplying $\tilde{v}$ by the appropriate correction factor, using (22), to obtain v^*.[39]

Illogical claims. One can employ the relationship between a patent's claimed rank and share of profit to impose logical consistency on valuation claims. For example, in a portfolio of 100 patents, it is logically possible that a patent is worth 20 times the average patent. But this valuation is logically possible *only if* the patent in question is ranked correctly.[40]

Figure 5 illustrates this logical constraint. The combination of a patent's claimed rank $\tilde{n}$ and claimed value relative to the mean patent $\tilde{K}_n$ may be divided into two regions: those that are logically possible, or AUC-consistent, and those that are AUC-inconsistent. According to equation (19), the dividing line is given by the value of M_n (Table 4), which reports the mean value of the top $(1-n)\%$ patents. Figure 5 shows that it is logically impossible, for example, for the

[38] The correction that follows can be extended in an obvious fashion if there exists still better information (3 or more known subsamples) on which patents have been included erroneously.

[39] To continue the prior example: suppose that $d = 0.40$ (so 40 of the 100 improperly added patents rank below the asserted patent, and 60 rank above it). The patent's true rank is then 110 - 60 = 50^{th} out of 1000, which is confirmed by $n^* = 0.90 - (0.10)(0.40 - 0.90) = 0.95$. According to Table 2, $K_{95} = 3.7$ and $K_{90} = 2.1$, so the patent's true share of portfolio value increases by a factor of $1.1 \times (3.7/2.1) = 1.94$, from $2.1/1100 = 0.19\%$ to $3.7/1000 = 0.37\%$.

[40] To see this, suppose that the patent ranks sixth (*i.e.*, 95^{th} percentile) in the portfolio. Then each of the five patents ranked above it must be worth at least 20 times the mean patent as well. Then their sum must be greater than 100 times the mean (or, equivalently, the sum of the K_n exceeds $P = 100$), which violates (10). Ultimately, these violations are traceable to a violation of the adding-up constraint (2).

95^{th}-percentile patent to be worth 20 times the mean, when all patents above the 95^{th} percentile average only 10 times the mean. Figure 5 also plots the median values for K_n (from Table 2), which shows that these values are, in fact, AUC-consistent.

Figure 5: Combinations of $(n,\ K_n)$ that are AUC-consistent and AUC-inconsistent

Uncertainty as to patent rank and portfolio size. More generally, assume that both n and P are unknown, but that a one-to-one mapping $n \iff K_n$ (such as that given by equation (17) and Table 2) is known, at least up to σ_v.

We focus here on the patentee's claimed share of total profit, C/V. To conform to the estimates reported in Table 2, a patentee's claim C must equal the infringer's average profit per patent, V/P, multiplied by K_n:

$$C = \frac{V}{P}\ K_n,\ \text{ or } \frac{C}{V} = \frac{K_n}{P} \tag{23}$$

Substituting (17) into (23) and rearranging, we have

$$n = F\left(\ln(\frac{C}{V}P)/\sigma_v + \sigma_v/2\right) \tag{24}$$

To discretize (24), we define a patent's rank r to be $r = \text{Int}\,[P(1-n)+1]$, so that (24) becomes

$$r = \text{Int}\,[P(1-F(\cdot))+1] \tag{25}$$

Figure 6a plots the relationship between r and P for claimed value shares C/V ranging from 0.1% to 10% (under the assumption that $\sigma_v = 1.71$), and for portfolios ranging in size from 10 to 1000 patents. Over this wide parameter space, the rank require to sustain a given claimed share of total value is remarkably stable—and the more so, as the claimed share increases.[41] Figure 6b plots the same relationships assuming $\sigma_v = 2.5$. One can readily see that, for the empirically relevant range of values for σ_v, the greater the skew of the distribution, the narrower the range of possible ranks for a given claimed share of total value, at least in the right tail.

[41] For example, if a patentee claims that his patent is worth 2% of total profit, then the patent must rank between 5^{th} and 11^{th} in the portfolio—regardless of portfolio size. If the claimed share is 4%, the rank must fall between 2^{nd} and 6^{th}.

Figure 6: Required patent rank given portfolio size P and claimed share of value C/V

Figures 6a and 6b greatly simplify the evidentiary problem that courts confront when a plaintiff brings a large damages claim against a complex product. This problem could potentially become mired in subsidiary disputes, such as computing the number of patents among which to apportion profit, and properly circumscribing the discovery of evidence regarding the value of these (otherwise unrelated) patents. Figures 6a and 6b show that, if the damages claim is "large," a plaintiff must be able to prove (or a defendant must be able to disprove) the rank of the asserted patent relative to a small subset of the portfolio. By analogizing to pleading requirements and other standards of proof, a court could schedule a threshold determination as to whether or not the plaintiff's damages claim, framed in terms of the share of value demanded and the patent's rank, possessed sufficient factual basis to proceed to a jury. In some cases, such as *Lucent v. Microsoft*, this determination could obviate the need for a trial and appeal, if it were clear from early stages of the litigation that the plaintiff lacked the facts to prove the ranking implied by his damages claim.

Departures from log-normality. A party may object that the patent value distribution is not lognormal, or that it comprises a mix of distributions, or that the draws are not independent. These general statistical properties obviously lie beyond the scope of this paper. But as a rule, a Lorenz graph's properties do not change markedly (relative to the typical differences of opinion between plaintiff and defendant economic experts), even with changes in the underlying statistical assumptions. This statement is generally true even in the northeast region of the graph, which is most likely to be of interest in litigation.

For example, construct a sample randomly drawn from a mixed empirical distribution comprising draws from uniform, normal, exponential and logit distributions, each having expected values randomly drawn from $U(a, b)$, and censor negative draws at zero to simulate the (erroneous) inclusion of zero-value patents. It is not difficult to construct lognormal approximations to such an empirical distribution, with errors weighted by patent value, to produce a Lorenz graph and ratios to the mean K_n that closely approximate the empirical distribution, as shown in Figure 7.

Figure 7: Lognormal approximation to an empirical distribution comprising 4 other distributions

5 Applications

5.1 *Oracle v. Google*

In a complex set of claims brought by Oracle against Google,[42] Oracle alleged (among other things) that Google infringed several patents related to Oracle's Java. After a series of pretrial expert reports and hearings, the trial court eventually narrowed the damages issue to the following assumptions: out of $P = 569$ patents that Oracle would have licensed to Google for $V =$ \$598 million, three allegedly infringed patents could reliably be ranked among the top 22 patents in the portfolio. Given those assumptions, the court asked: what was the value of one asserted patent?[43]

To address this question, Oracle's expert cited three surveys of patent value. Unlike the studies summarized in Table 1, none of these was a large-sample analysis derived from optimizing behavior. Based on the average of inferences from these surveys, Oracle's expert concluded that the top 22 patents were worth about 77% of the \$598 million whole, or about \$21 million per patent (about 20 times the mean).

The estimates reported in Table 3 indicate that this claim is likely overstated: the top 3.9% of patents account for about 46% of the total, or an average per patent of about 11.9 times the mean (from the composite medians reported in Table 1); 11.9 $\times$ \$598 million / 569 patents = \$12.5 million per patent. Assuming a plausible range for σ_v of [1.5, 2.2], the range of values is about \$10.6–18.0 million per patent.

For its part, counsel for Google denied that *anyone* could solve the problem framed by the trial judge, at least based on the kind of large-sample evidence of the kind cited by Oracle (and

[42] No. 3:10-cv-03561-WHA, N. D. California; the proceedings are reported comprehensively at http://www.groklaw.net/staticpages/index.php?page=OracleGoogle. The author consulted for the court-appointed expert on damages issues.

[43] Order Granting in Part and Denying in Part Google's *Daubert* Motion to Exclude Dr. Cockburn's Third Report, March 13, 2012, Dkt. No. 785.

reported in Table 1).[44] The present study shows that such broad criticisms are misplaced: for many purposes, study-specific differences across technologies, countries and methodological assumptions can be reduced to variations in a single parameter that falls within a relatively narrow range.

Suppose instead that the 569 patents that Oracle would have licensed to Google represented the most valuable of a set of 2,200 Java-related patents, so the sample selection factor m is about 0.74. Then the top 22 patents represent the top 1% of this superset P^+, having an average value of about 25.4 times the mean (Table 4), or about twice the ratio to the mean of the observed sample P. But in the right tail sample selection works against the plaintiff: by analogy to (20), the mean value of P^+ is $1/M_{0.74} \approx 0.3$ times the mean of P; the larger sample's lower mean value more than offsets the asserted patents' relatively higher rank, yielding an expected value of \$8.1 million per patent, about 65% of the original estimate (Figure 4).

5.2 *Lucent v. Microsoft*

Often, the problem facing the analyst is not framed as neatly as in *Oracle v. Google*, perhaps because the number of patents P, and/or the rank of the asserted patent n, is unknown or subject to dispute. But even in such circumstances, one can still evaluate alleged economic relationships for plausibility and logical consistency. In particular, it is important to distinguish between claims that are merely unlikely from those that are logically impossible.

Recall that the *Lucent* jury awarded Lucent about \$358 million. To simplify the exposition, we assume that this is Lucent's claim C. Under the conservative assumptions that Microsoft's incremental profit margin was 40% of its \$8 billion in accused Outlook sales, and that all of Microsoft's profit should be allocated to patents (so V = \$3.2 billion), Lucent's claim represents

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> Unfortunately, all of those studies are inapposite, looking at the broadest possible context—a random sampling of all patents, owned by all patentees, related to all technology areas. None of the studies looked at a single party's narrow patent portfolio covering only one technology area, like the [Oracle] portfolio at issue here. Essentially, [Oracle's expert] was given a fruit basket and asked to estimate the percentage of the basket's value attributable to the apples in the basket. For some reason, he attempted to answer the question by looking at the distribution of value among all groceries in every department in the supermarket.

Google's Reply in Support of Motion to Strike Portions of Third Expert Report by Iain Cockburn and Expert Report by Steven Shugan, February 28, 2012, Dkt. No. 751.

about $C/V = 11.2\%$ of Microsoft's profit.

Using (25), we can trace the regions that are consistent with Lucent's claim, for any combination of r and P. As the assumed complexity of the accused Outlook product increases—where "complexity" means the number of patents it embodies—the higher the asserted Lucent patent must rank, to justify Lucent's claim. For example, Figure 6 implies that Lucent's invention must have ranked no lower than second, as long as $10 \leq P \leq 1000$. This conclusion is inconsistent, of course, with the Federal Circuit's decision that the Lucent invention could not constitute more than a "tiny fraction" of the total value of Outlook.

In short, Lucent's claim could have been rejected on summary determination, rather than after a trial and appeal, as long as the trial judge framed the dispute in terms of the patent's rank and agreed that a rank of first or second was inconsistent with any interpretation of the evidence.

While it may be difficult to determine P precisely, the foregoing analysis produces other economic implications that are also subject to verification. For example, assume that $P = 100$: Microsoft's \$3.2 billion in (US) profit implies an average of \$32 million per patent, if its patents caused all profit. This average is, to say the least, highly unusual for a large sample of patents, particularly in the information technology sector.[45] But as equation (24) makes clear, if one increases the number of relevant Microsoft patents, or reduces the portion of V that is attributable to patents, to obtain a more plausible average, then Lucent's claim represents an even larger share of the whole, and must rank even higher to satisfy the constraint (23).[46]

[45] One catalogue of 31 IT-related patent portfolio transactions from 2010 through 2013 totaled close to \$24 billion for about 45,000 inventions, an average of about \$524,000 (worldwide) per invention (median: \$330,000). *High-Visibility Patent Sales & IP-Driven M&A Transactions*, Ronald Laurie, Inflexion Point Group, available at: http://www.law.berkeley.edu/files/Panel_20_Laurie.pdf

Moreover, if the average Microsoft invention were worth \$32 million, then Microsoft's 18,000 patented inventions would be worth \$576 billion in the US alone, far exceeding Microsoft's then-current market capitalization.

[46] Under the somewhat more plausible assumptions that: (a) Outlook comprises 500 patents, (b) half of Outlook's profit is attributable to these patents, and (c) Lucent's patent ranked in the 95^{th} percentile, Lucent's claim should have been less than than \$3.2 million $\times$ 3.7 = \$12 million.

5.3 Standard-essential patents

In recent decisions,[47] federal judges have set royalty rates for patents they have determined to be essential to a technical standard. This exercise may require determining the patents' rank and the share of value appropriate to that rank. For example, in *In re Innovatio,* the judge relied on a non-economic study of electronics patents to conclude that the top 10% of patents accounted for 84% of total value. According to Table 4, a better estimate is about 64%, though parameters supporting the higher figure have been reported in certain studies.

But the larger question is whether the assumption of a linear adding-up constraint is appropriate to patent portfolios that exhibit strong complementaries.[48] This property poses especially difficult valuation challenges, since patents that are strong *complements in production* (*i.e.,* when used together) are likely to be strong *substitutes in litigation* (*i.e.,* when excluding a rival's use: if the infringement of any perfectly complementary patent is sufficient to foreclose competition, then one complement is as good as another at effecting foreclosure). Under such circumstances, the assumption of a linear adding-up constraint may yield less accurate results.

5.4 Global litigation incentives

The recent "smartphone patent wars" have occasioned an "explosion" of litigation, which threatens to disrupt global trade patterns. Frequently accused infringers have proposed that patentees be prevented from enforcing their right to exclude, their recovery being limited to a royalty.[49] The

[47] *Microsoft Corp. v. Motorola Inc.,* C10-1823JLR, (W.D. Wash. Apr. 25, 2013); *In re Innovatio IP Ventures,* MDL 2303, No. 975 1:11-cv-09308.

[48] Complementaries may also arise outside of patents that are not necessarily essential to any technical standard. For example, in *Oracle v. Google,* the court cited Oracle's economist in observing that, "patents in a single portfolio derive value from complementing each other to prevent design around, meaning that unasserted patents are valuable because they prevent design around asserted patents." ORDER GRANTING IN PART AND DENYING IN PART GOOGLE'S *Daubert* MOTION TO EXCLUDE DR. COCKBURN'S THIRD REPORT, March 13, 2012, Dkt. No. 785.

[49] "Verizon thinks it would be great if President Obama, in a blanket statement, made clear he would not let stand any decision blocking importation of consumer wireless devices. The parties then would have to recur to normal patent litigation, and whatever rights and wrongs are discovered could be settled by exchanges of cash." H. W. Jenkins, "Obama and the Smartphone Wars," *The Wall Street Journal*, August 24, 2011, available at: http://online.wsj.com/article/SB10001424053111903327904576526130093390612.html.

In August 2013, the Obama administration "disapproved" (*i.e.,* formally overturned) a decision of the International Trade Commission to block the importation of certain infringing devices manufactured by Apple. Letter from Michael Froman, US Trade Representative, to Irving Williamson, Chairman of the US International Trade Commission question then becomes: under the assumptions of this paper, what sort of royalty should these parties expect?

I collected basic financial information, including annual R&D and sales, on 17 firms with significant telecommunications-related R&D and/or products.[50] Over the period 2000-2010, the mean firm averaged about \$34.6 billion in sales, about 5.5% of which was spent on R&D. Assuming a 15% depreciation rate (Lanjouw *et al.* 1998), these figures implies a steady-state R&D stock of about 37% of sales, or about \$12.8 billion.

Data from the US Patent and Trademark Office indicate that these firms obtained an average of 599 US patents per year (which we take to proxy for their total worldwide inventions), or an average stock (after accounting for dropouts) of about 5,324 inventions over this period. Assuming conservatively that 100% of a firm's R&D stock is appropriable via the exclusionary power of its patented inventions,[51] this implies that the stock of R&D is worth about \$2.4 million per invention.[52] Thus, according to Table 2, a 99^{th}-percentile telecom invention's global value is conservatively expected to be worth about \$2.4 million $\times$ 12.0 = \$28.9 million.

Under the further conservative assumptions that: (1) 50% of global patent rights are attributable to the US,[53] (2) an accused infringer accounts for 50% of the US market, and (3) the infringement period accounts for 50% of the patent's useful life, the portion of the invention's total global value that one would expect to be awarded in a US litigation is about \$3.6 million.[54] It is important to emphasize that this relatively small amount, when aggregated over all inventions, countries, time periods and competitors, generates the mean firm's \$12.8 billion R&D stock.

(August 3, 2013), available at: http://www.ustr.gov/sites/default/files/08032013%20Letter_1.PDF.

[50]The 17 firms are: Siemens, AT&T, Samsung, LG Electronics, France Telecom, Nokia, NTT DoCoMo, Motorola, Ericsson, Alcatel-Lucent, Apple, Nortel, Kyocera, Qualcomm, Research in Motion, Broadcom, and InterDigital.

[51]Lanjouw *et al.* (1998) report that national patent rights represent 15-25% of national R&D expenditures. Various surveys, beginning with Levin *et al.* (1987), have found that patent rights are not the most effective appropriability mechanism, except in a handful of industries.

[52]These figures imply an average R&D cost of about \$3.2 million per patented invention. The mean R&D cost per invention (a flow variable) differs from the mean R&D stock per invention because the annual number of patents appears to depreciate more slowly than the assumed depreciation rate for R&D (15% per year).

[53]Depending on the product market, US telecom sales account for approximately 25% of the global total. However, US *patent rights* are almost certainly worth more than the global average, after controlling for market size (Putnam 1996).

[54]For obvious reasons, this expected value does not represent the value to be calculated in any particular litigation over any particular patent.

To set this figure in context, US patent litigation costs an average of $1.6 million (through discovery) to $2.6 million (through trial) when the amount in controversy is between $1 and $25 million (AIPLA 2011). Litigation thus consumes a high fraction of the royalty payment that is to be expected when an adding-up constraint is imposed, and is therefore relatively inefficient. It is easy to see why a plaintiff might seek to increase the yield on its investment in litigation, by claiming "above average" compensation not tied to any known average.

More generally, litigation over complex devices, which generally limits a patentee to asserting no more than a handful of patents in any one case, is likely to be a relatively inefficient method of recouping a firm's R&D investment from its rivals, unless it can show that the asserted patent accounts for a large fraction of the realized value of that investment. Policymakers who wish to decrease "abusive" litigation should also be concerned with the R&D disincentives caused by introducing additional inefficiencies into private efforts to transact patent portfolios that cover technologically complex products, when the enforcement of any individual patent proxies poorly for the value of the portfolio as a whole.[55]

6 Conclusion

In expectation, fully informed agents who exchange goods voluntarily create fair market values.[56] Such is not the case in non-market situations, like litigation, where there is no "exchange," and compensation is compelled by court order. Anticipating that compulsion, litigating parties often make claims strategically, obscuring or suppressing market-based evidence, constrained only tenuously by the framework of a "hypothetical negotiation" between them. The costs of strategic behavior are not limited to non-market settings: strategy-induced failures to reach a welfare-improving bargain (influenced, perhaps, by the threat of litigation-based compensation rules) reduce marketplace efficiency as well. These problems are greatly exacerbated in technologically complex products that may embody hundred of patents, only a few of which can be scrutinized at a time.

[55]For example, in litigation over standard-essential patents, defendants sometimes ask courts to set a "FRAND royalty rate," in lieu of granting an injunction. But even if this rate is determined accurately, it only resolves the dispute as to the asserted patents, not the patentee's (typically much larger) portfolio of essential patents. If a patentee must litigate the entire portfolio *seriatim*, most of the returns to patent protection are dissipated.

[56]Fair market value is "the price at which the property would change hands between a willing buyer and a willing seller, neither being under any compulsion to buy or to sell and both having reasonable knowledge of relevant facts." *United States v. Cartwright*, 411 U.S. 546 (1973).

This paper provides some simple, broadly applicable guidelines for translating aggregate market data (product profit or portfolio license fees) into valuations of the individual patents that cause that value. The primary assumption underlying these guidelines is that the parts must add up to the whole. The primary input into their application is a patent's rank among the parts. Together, these assumptions ensure that not all patents are "above average."

Because these guidelines depend on the determination of a patent's relative rank, which may be a fact-intensive inquiry, they are not simply "a rule of thumb." Nor do they suffer from the criticism that they "fail to tie [the apportioned value] to the facts of the case."[57] Rather, the guidelines place the focus where it should be: on using all available information to rank each patent's value against the value created by other patent and non-patent inputs. In addition to ruling out logical impossibilities, such rankings also assist negotiating parties (and triers of fact) in formulating logically consistent valuation claims, and in determining the likelihood that such claims are true.

Because I adopt a very general framework, the reported results are robust to various specification errors and other violations of the assumptions. For example, all the results reported in Section 2.1 are entirely non-parametric, so the assumption that patent values are distributed log-normally is not necessary. Similarly, the results are also robust to some dependence between observations, as might occur (for example) if firms patent improvements on (or otherwise build fences around) especially valuable patents. Finally, because of the similarity of σ_v across a wide variety of technologies, the results do not depend on how "comparable" is the technology or industry of the asserted patent to any particular study in the literature.

In light of the factors used by courts to evaluate "scientific methods,"[58] it is worth summariz-

[57]For decades, US courts accepted a so-called "25 percent rule of thumb" when apportioning profit. Under this "rule," a patentee and an infringer are assumed to agree on a baseline royalty equal to 25% of the accounting (operating) profit of an infringing product, with the remaining profit attributable to other inputs. This baseline assumption may be adjusted up or down in an *ad hoc* fashion, based on other evidence. The author of the rule justified its use partly on the grounds that "complicated mathematical formulae or analyses . . . are likely to lose the interest of judge or jury, and would probably never have been employed by the parties in reaching a decision about whether or not to infringe a patent" (Goldscheider 1996).

The Federal Circuit recently held the 25 Percent Rule inadmissible as a matter of law, because "it fails to tie a reasonable royalty base [*sic*] to the facts of the case." *Uniloc USA, Inc. v. Microsoft Corp.*, 632 F.3d 1292 (Fed. Cir. 2011).

[58]In deciding whether to admit proposed expert testimony, Rule 702 of the Federal Rules of Civil Procedure mandates that:

1. the testimony be based upon sufficient facts or data

ing the evidentiary basis for conclusions drawn from the paper's results. The theory described here is derived from the properties of statistical distributions, and some elementary calculus. These are, of course, widely accepted.[59] The parameters of that theory have been identified from a summary of widely cited empirical studies, which propose and test models of the patent value distribution under diverse circumstances. The range of those parameters, as well as the variability they induce in finite samples, both imply that real-world patent portfolios will deviate to some extent from expectations. Fortunately, both theory and empirical investigation show that the error rate is both known and relatively small (particularly for patents ranked below $v_{\max}$), the more so as portfolio sizes increase. Obvious misuses of the data can be corrected relatively simply. And, perhaps surprisingly, the evaluation of a complex valuation claim often can be reduced to determining a patent's absolute rank within a portfolio, with little regard for the size of that portfolio.

There remains significant work to be done. The causal linkage between patents and economic profit, the proof of that linkage in negotiation and litigation, and the relationship of that linkage to conventional accounting measures of profit, all demand further economic investigation. Fortunately, US courts now seem generally willing to entertain studies aimed at satisfying that demand.

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2. the testimony be the product of reliable principles and methods
3. the witness have applied the principles and methods reliably to the facts of the case

In deciding whether a proposed principle or method is sufficiently reliable, the US Supreme Court has defined "scientific methodology" as the process of formulating hypotheses and then conducting experiments to prove or falsify the hypothesis. The trial judge must then weigh the following factors in establishing a method's validity:

- Empirical testing: is the theory or technique falsifiable, refutable, and testable.
- Whether or not the theory been subjected to peer review and publication.
- The existence of a known or potential error rate.
- The existence and maintenance of standards and controls concerning the method's operation.
- The degree to which the theory and technique is generally accepted by a relevant scientific community.

Daubert v. Merrell Dow Pharmaceuticals, 509 U.S. 579 (1993).

[59]It should be noted that the trial judge in *Oracle v. Google* (who apparently holds an undergraduate degree in mathematics) refused to admit testimony on Nash bargaining, after quoting a simple description of the first-order conditions for profit maximization familiar from intermediate microeconomics texts:

> No jury could follow this Greek or testimony trying to explain it. The Nash bargaining solution would invite a miscarriage of justice by clothing a fifty-percent assumption in an impenetrable façade of mathematics.... Instead, the normal Georgia-Pacific factors, ... which are more understandable to the average fact-finder, will guide our reasonable royalty analysis.

Order Granting in Part Motion to Strike Expert Report of Plaintiff Expert Iain Cockburn, July 22, 2011, Dkt. No. 230. It should go without saying that the scientific validity of a theory is independent of a jury's ability to "follow this Greek," which depends instead (for economics no less than other scientific disciplines) on the ability of counsel to present complex testimony simply and comprehensibly to the "average fact-finder."

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Table 1
Value of σ_V implied from various patent renewal and patent application studies

	Patent renewal models																Patent application models						
Study	Schankerman and Pakes (1986)			Pakes (1986)			Sullivan (1994)	Lanjouw (1998)				Schankerman (1998)				Bessen (2008)	Putnam (1996)		Chan (2010)	Deng (2011)			
Country	Germany	France	U.K.	Germany	France	U.K.	U.K.	Germany				France				U.S.	Germany	World	Various	Europe			
Year	1952-78	1951-79	1950-76	1952-72	1951-79	1950-74	1852-76	1953-80				1969-82				1991	1974		1990-2000	1978-96			
Technology	All	All	All	All	All	All	All	Comp	Textile	Engines	Pharma	Pharma	Chem	Mech	Elec	All	All	All	Biotech	Pharma	Elec		
Initial return	LN	LN	LN	LN	LN	LN	LN	E	E	E	E	LN	LN	LN	LN	LN	LN	LN	LN	LN	LN		
Returns evolution	D	D	D	S	S	S	D	S	S	S	S	D	D	D	D	D	D	D	D	S	S		
Quantile																						Median	Mean
0.25 - 0.50	1.71	3.04	2.07	1.69	2.91	2.15	1.61	-	-	-	-	1.71	1.89	2.26	2.52	1.85	1.91	2.18	3.26	-	-	2.08	2.21
0.50 - 0.75	1.57	2.31	1.73	1.69	2.88	2.46	1.69	1.33	1.49	1.03	1.44	1.78	1.92	2.29	2.64	1.85	2.73	1.96	3.23	2.31	2.20	2.06	2.13
0.75 - 0.90	1.54	2.02	1.64	1.34	2.54	1.69	1.73	1.03	1.13	0.90	1.15	1.28	1.42	1.79	2.13	1.85	2.60	1.85	3.22	2.42	2.25	1.79	1.93
0.90 - 0.95	1.46	1.92	1.56	1.09	1.64	1.23	1.57	0.92	0.98	0.80	0.99	1.44	1.58	1.98	2.36	1.85	2.23	1.76	2.83	2.28	2.20	1.70	1.72
0.95 - 0.99	1.46	1.92	1.56	0.86	1.09	0.92	1.67	0.80	0.87	0.71	0.82	1.41	1.55	1.97	2.37	1.85	1.98	1.74	2.76	2.24	2.34	1.71	1.62

Notes and sources:

1. "Initial return" refers to the assumption that the invention's initial draw follows a log-normal (LN) or exponential (E) distribution.
2. "Returns evolution" refers to the deterministic (D) or stochastic (S) process described in the text.
3. Bessen (2008) does not report value estimates, so the median value of σ for the initial returns distribution r_1 is reported instead.
4. To avoid over-weighting those studies that report estimates for individual technology fields (Lanjouw 1998, Schankerman 1998, Deng 2011) or countries (Chan 2010), the study estimates were first averaged, then the study average was used in calculating the median and mean across studies.

Table 2

K_n: the ratio of the n^{th} percentile patent value to the mean patent overall, for selected σ_v

	Value of σ_v												
n	0.5	1.0	1.2	1.4	1.5	1.6	1.8	2.0	2.2	2.5	2.8	3.0	Median[1]
0.25						0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.1
0.50						0.3	0.2	0.1	0.1	0.0	0.0	0.0	0.2
0.75						0.8	0.7	0.5	0.4	0.2	0.1	0.1	0.8
0.90				2.3	2.2	2.2	2.0	1.8	1.5	1.1	0.7		2.1
0.95				3.8	3.8	3.9	3.8	3.6	3.3	2.7	2.0		3.7
0.98			5.7	6.7	7.1	7.4	8.0	8.2	8.2	7.5	6.2		7.6
0.99		6.2	7.9	9.7	10.6	11.5	13.0	14.2	14.9	14.7	13.4		12.0

Notes and sources:

1. "Median" refers to the median of the estimates reported in Table 1.
2. A blank cell for a given pair (σ_v, n) implies that none of the studies summarized in Table 1 reported values for that parameter combination.

Table 3

K_l^u: the ratio of the value of the mean patent in (*l, u*) to the mean patent overall

	Upper limit *u*							
Lower limit *l*	0.25	0.50	0.75	0.90	0.95	0.98	0.99	Max
0.00	0.0	0.1	0.2	0.4	0.5	0.7	0.8	1.0
0.25		0.2	0.3	0.5	0.7	0.9	1.0	1.3
0.50			0.4	0.8	1.0	1.2	1.4	1.9
0.75				1.3	1.7	2.1	2.5	3.4
0.90					2.7	3.7	4.4	6.5
0.95						5.3	6.4	10.2
0.98							9.6	17.5
0.99								25.4

Notes and sources:

1. Values are computed using the median of the estimates reported in Table 1.

Table 4

M_n: the ratio of the mean value of patents above the n^{th} percentile to the mean patent overall, for selected σ_v

	Value of σ_v												
	0.5	1.0	1.2	1.4	1.5	1.6	1.8	2.0	2.2	2.5	2.8	3.0	Median[1]
n													
0.25						1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3
0.50						1.9	1.9	2.0	2.0	2.0	2.0	2.0	1.9
0.75						3.3	3.5	3.6	3.7	3.9	3.9	4.0	3.4
0.90				5.5	5.9	6.2	7.0	7.6	8.2	8.9	9.4		6.5
0.95				8.0	8.8	9.6	11.2	12.7	14.2	16.0	17.5		10.2
0.98			9.7	12.7	14.3	16.1	19.8	23.7	27.7	33.4	38.5		17.5
0.99		8.9	12.6	17.2	19.9	22.8	29.3	36.5	44.2	56.2	67.5		25.4

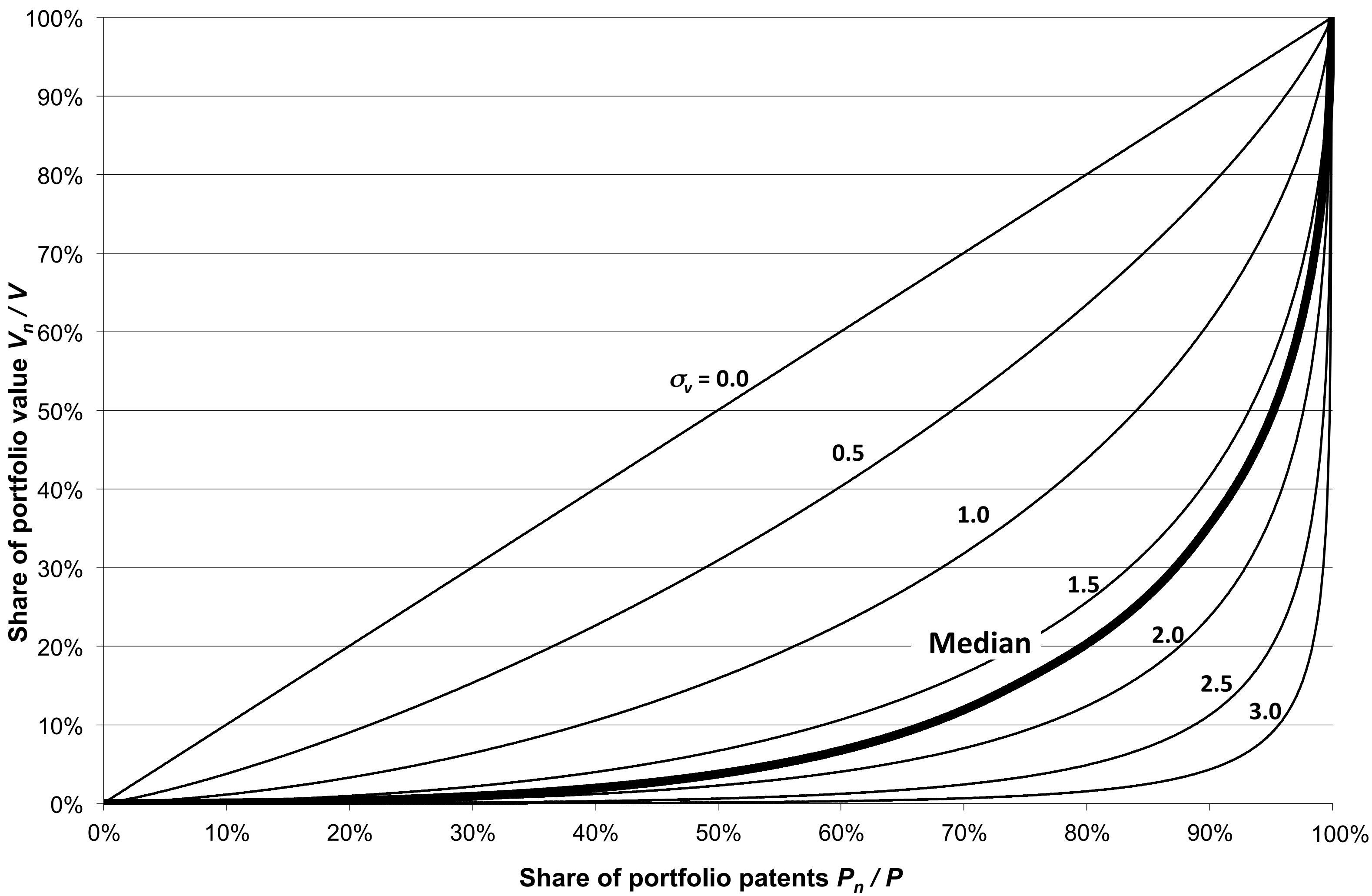
Notes and sources:

1. "Median" refers to the median of the estimates reported in Table 1.
2. A blank cell for a given pair (σ_v, n) implies that none of the studies summarized in Table 1 reported values for that pair.

Table 5
Calculation of a litigation claim for a 99th-percentile patent in a large telecommunications portfolio

Variable Name	Parameter	Value	Source
R&D stock			
Annual sales	S	\$34,628	ThomsonOne
Annual R&D	R	\$1,914	ThomsonOne
R&D intensity	r	0.055	$= S / R$
Annual depreciation rate	d	0.15	Lanjouw et al. (1998)
Implied R&D stock		\$12,757	$= R / d$
Patent portfolio value			
Patent rights share of R&D stock	π	1.00	Assumption
Aggregate value of patent rights		\$12,757	$= \pi R / d$
Number of patents	P	5,324	USPTO
Mean R&D stock per patent		\$2.40	$= \pi (R / d) / P$
Value attributable to an individual patent			
Percentile rank of n^{th} patent		0.99	Assumption
Ratio of n^{th}-percentile patent to mean patent	K_{99}	12.07	Table 2, σ =1.71
Global value of n^{th}-percentile patent	v_{99}	\$28.92	$= K_{99} \pi (R / d) / P$
Damages claim in a U.S. litigation			
Global patent rights attributable to U.S.	s_i	0.50	Assumption
Share of sales made in by infringer	s_i	0.50	Assumption
Infringement period as a percent of patent's useful life	s_t	0.50	Assumption
Value of damages claim	v_{99iit}	\$3.61	$= (s_i s_i s_t) K_{99} \pi (R / d) / P$
Litigation costs through discovery		\$1.60	AIPLA Economic Survey 2011
Litigation costs through trial		\$2.60	AIPLA Economic Survey 2011

Figure 1
Lorenz graphs of the patent value distribution, for selected σ_v



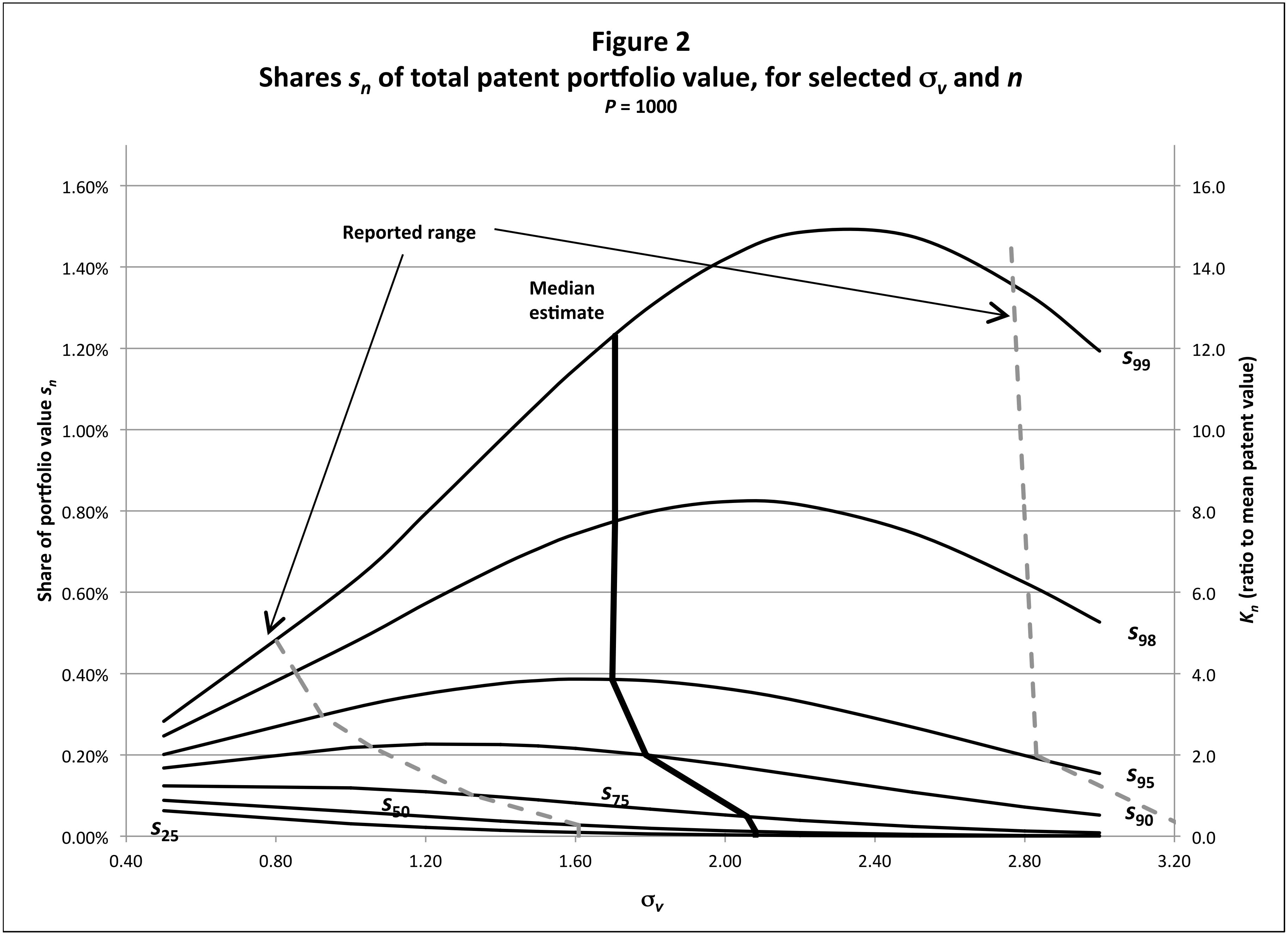


Figure 2
Shares s_n of total patent portfolio value, for selected σ_v and n
P = 1000
Reported range
Median estimate
Share of portfolio value s_n
K_n (ratio to mean patent value)
σ_v
1.60%
1.40%
1.20%
1.00%
0.80%
0.60%
0.40%
0.20%
0.00%
16.0
14.0
12.0
10.0
8.0
6.0
4.0
2.0
0.0
0.40
0.80
1.20
1.60
2.00
2.40
2.80
3.20
s_{99}
s_{98}
s_{95}
s_{90}
s_{75}
s_{50}
s_{25}

Figure 3
95% confidence intervals for $\bar{K}_n$ relative to the expected value, for selected *P*

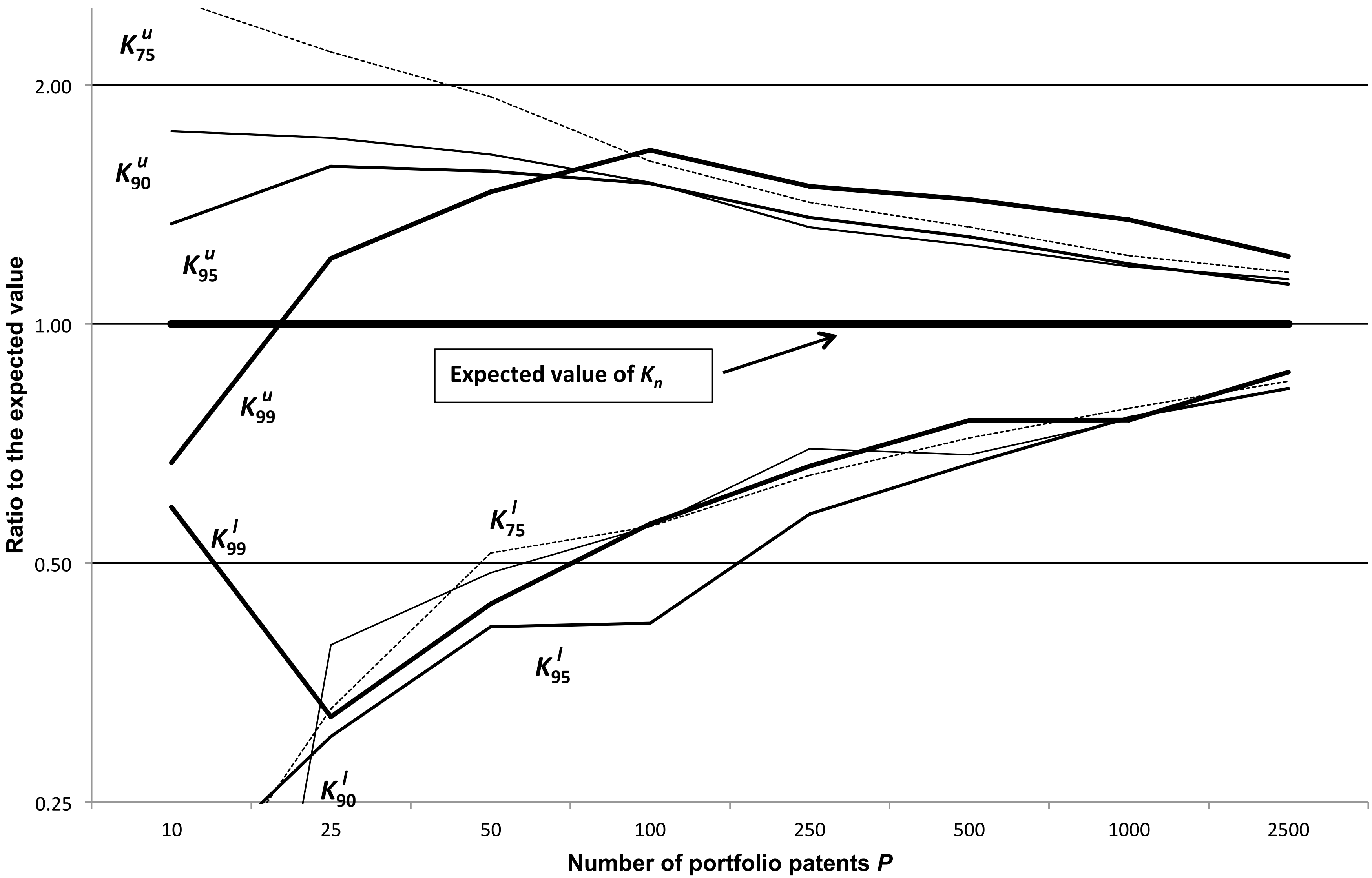


Figure 4

Ratio of $K^*_{n|m}$ to $K_{n|m}$ for selected n and m

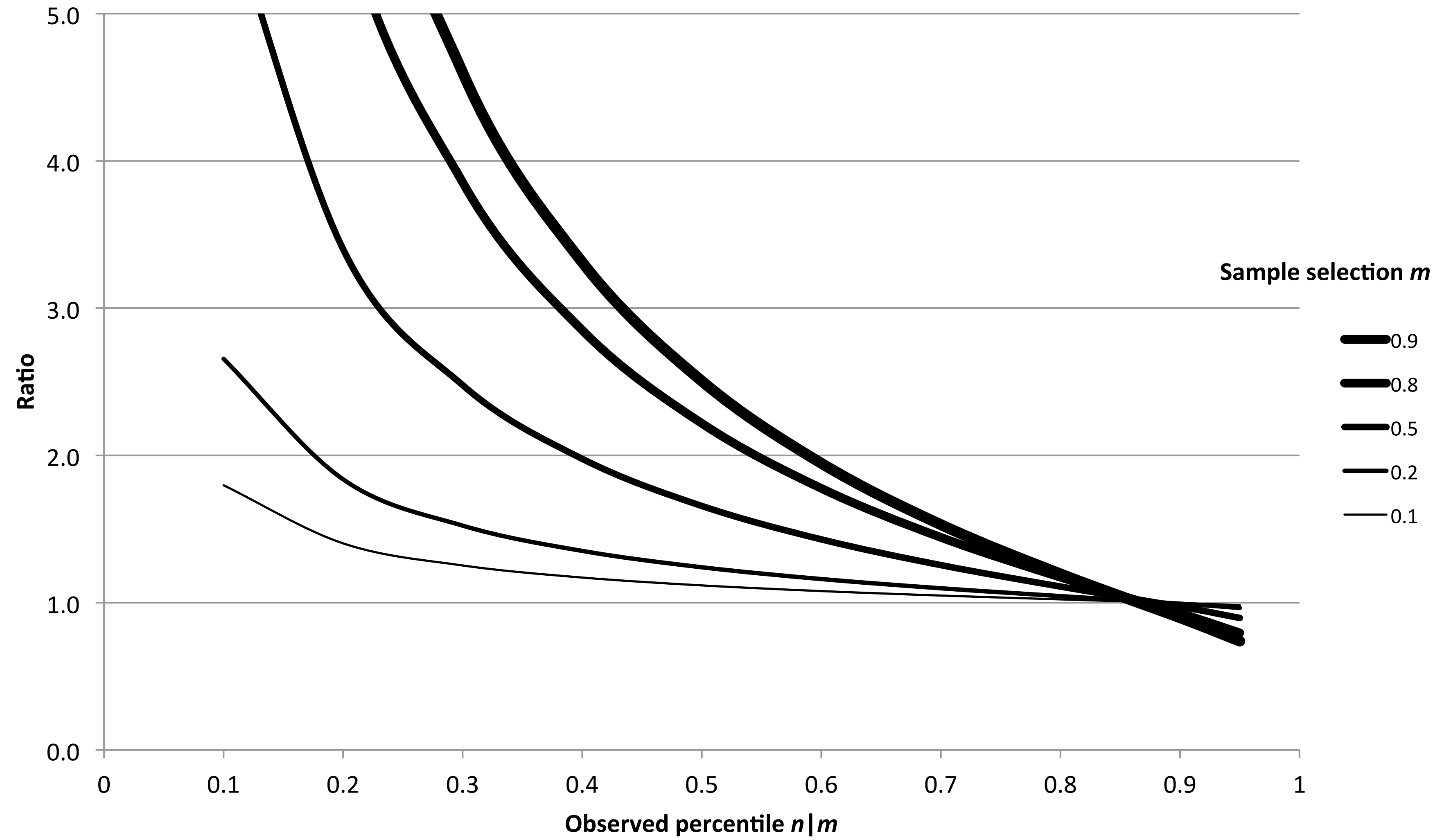
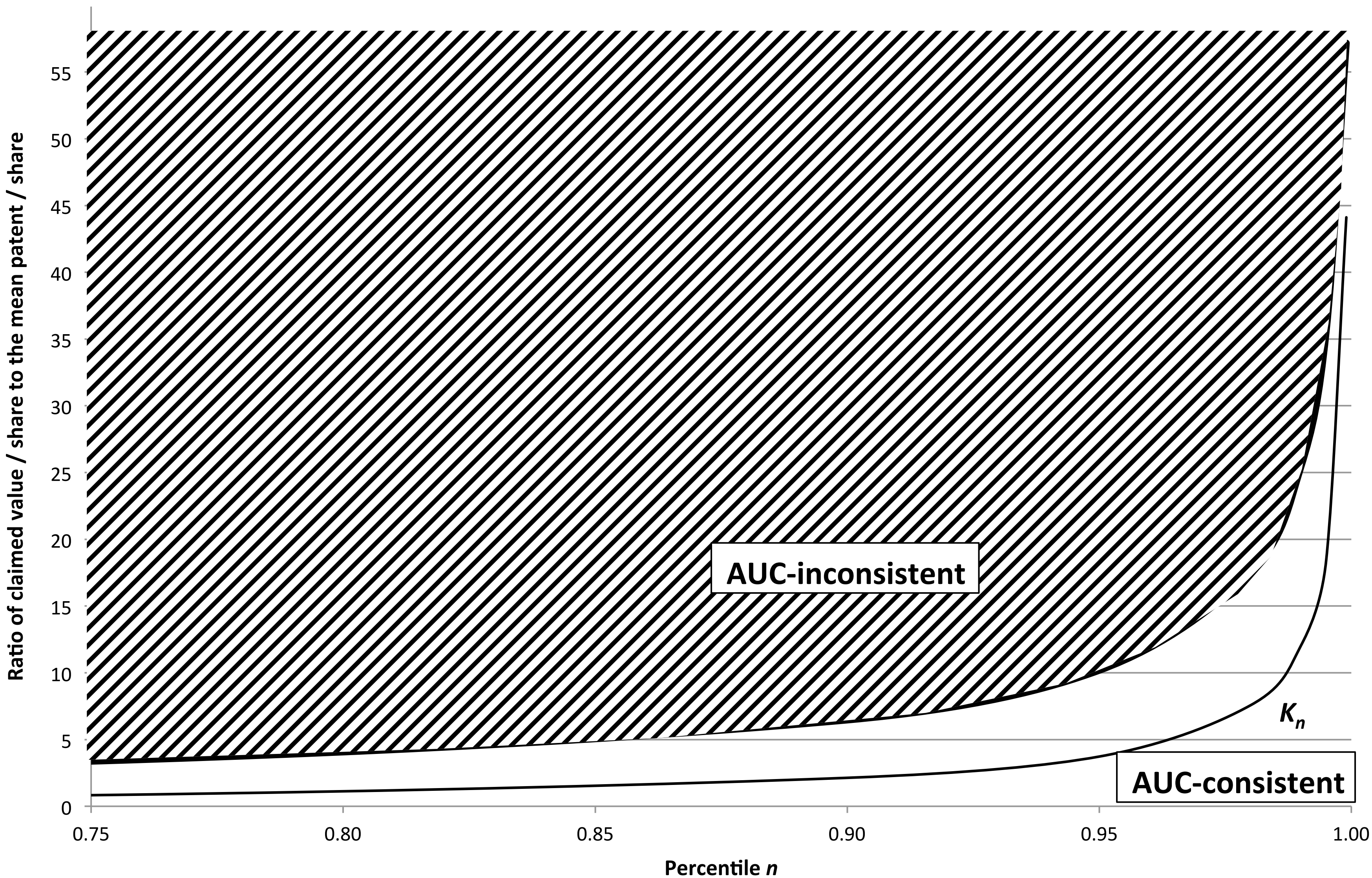


Figure 5
Combinations of (n, K_n) that are AUC-consistent and AUC-inconsistent



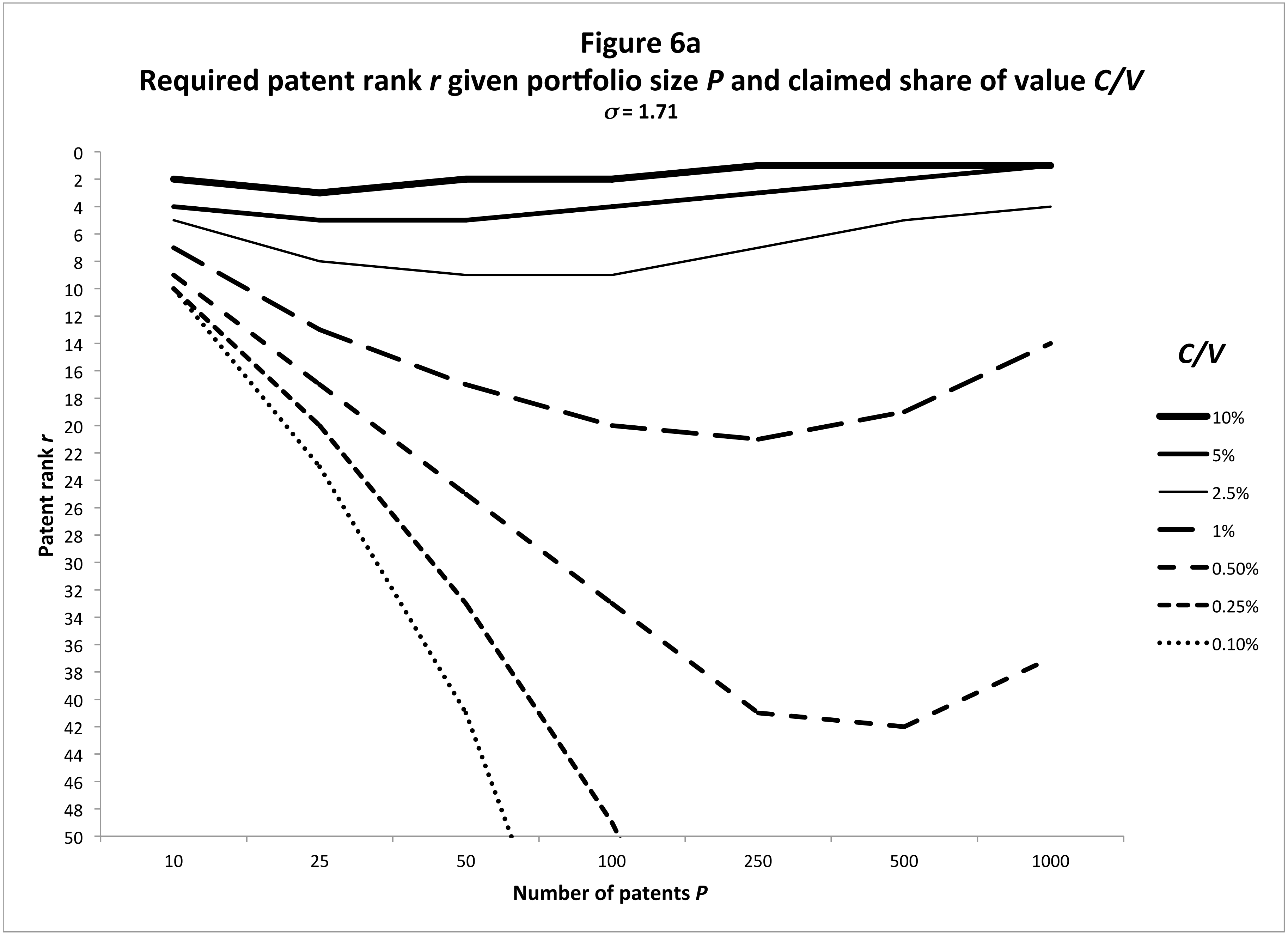


Figure 6a
Required patent rank r given portfolio size P and claimed share of value C/V
σ = 1.71
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
46
48
50
Patent rank r
10
25
50
100
250
500
1000
Number of patents P
C/V
10%
5%
2.5%
1%
0.50%
0.25%
0.10%

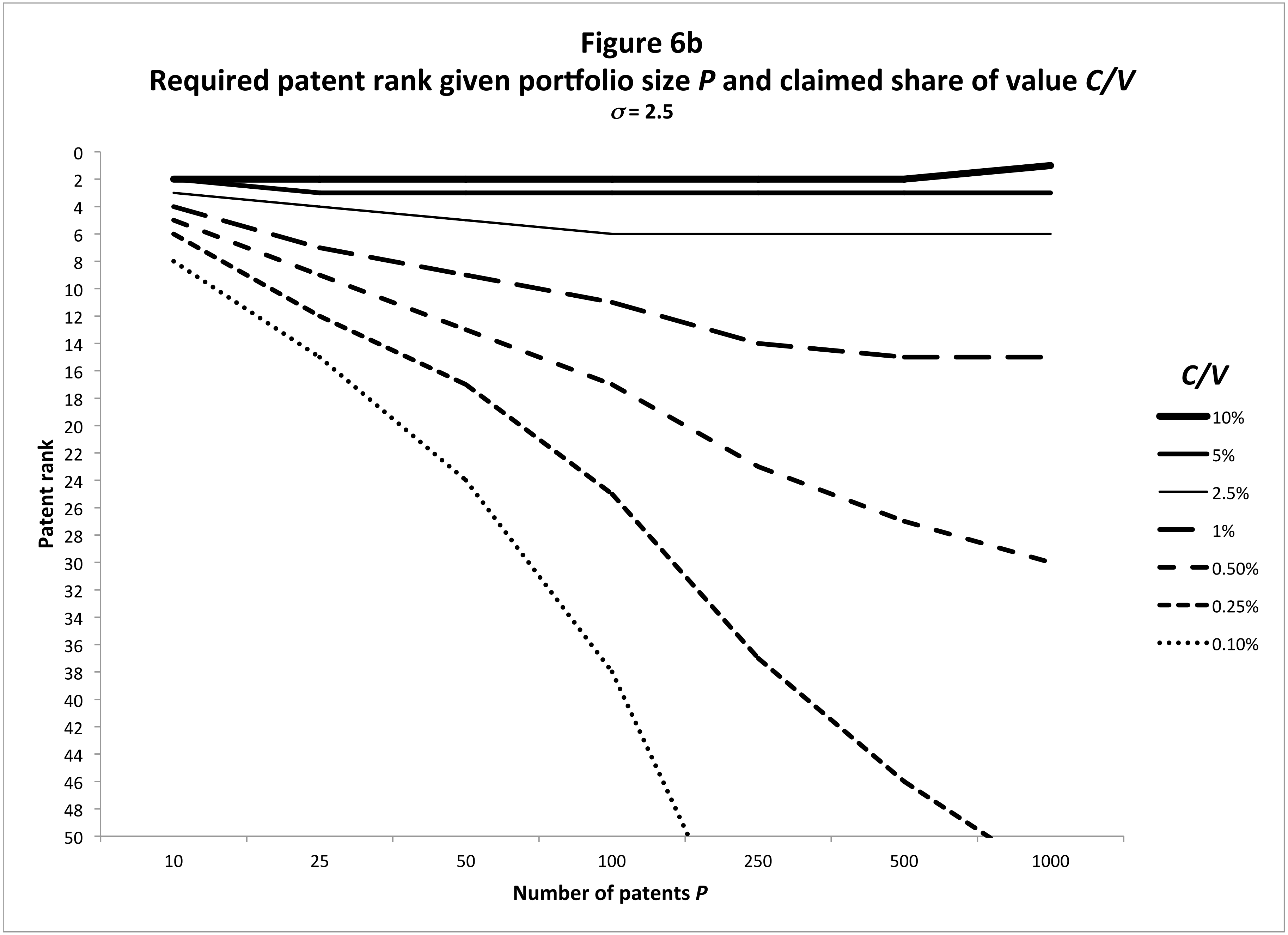


Figure 6b
Required patent rank given portfolio size P and claimed share of value C/V
σ = 2.5
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
46
48
50
Patent rank
10
25
50
100
250
500
1000
Number of patents P
C/V
10%
5%
2.5%
1%
0.50%
0.25%
0.10%

Figure 7
Lorenz graph for a mixed distribution and its closest log-normal approximation